Final Exam Topics
Math 24 Winter 2012

I. Vectors

Axioms of a vector space, vector addition, scalar multiplication.
Key examples: $\mathbb{R}^n$, $\mathrm{Mat}_{n,m}$, $\mathcal{P}_n(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$, $\mathrm{Fun}(S)$, $\mathrm{Fun}(S,V)$, $\mathcal{L}(V,W)$, $\{\mathbf{0}\}$.
Basic tasks: show something is or is not a vector space.
show properties of spaces by proof from the axioms.
Other "special spaces": subspaces, $\mathcal{L}(E)$ (aka $\mathrm{Span}(E)$), subspace sum and intersection.
Linear (in)dependence, basis, dimension, coordinates.
Paring down spanning sets, building up linearly independent sets.
Tools: linear dependence means some vector is dependent on earlier vectors.
subspace of equal dimension to full space is full space.
$\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$.

II. Linear Transformations and Matrices

Linearity: key tool for proofs (translate linear combinations between domain and range).
shows subspaces map to subspaces, gives linear extension (below).
Relationship between matrices and transformations including arithmetic and composition.
Key examples: identity, zero, scalar multiplication, differentiation, evaluation map, projection, rotation, reflection.
Matrix traits: diagonal, triangular, idempotent, symmetric, nilpotent.
Linear extension: action on any basis completely determines transformation.
Any action on a basis is allowed.
Compare: free choice in numbers placed into matrix.
For $T : V \to W$, $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{Im}(T))$.
Injectivity equivalences: image of basis is linearly independent set; $\ker(T) = \{\mathbf{0}\}$; columns of matrix are linearly independent; at most one solution to $A\boldsymbol{X} = \boldsymbol{B}$ for any $\boldsymbol{B} \in W$.
Surjectivity equivalences: image of basis spans; columns of matrix span; at least one solution to $A\boldsymbol{X} = \boldsymbol{B}$ for any $\boldsymbol{B} \in W$.
When $\dim(V) = \dim(W)$, injectivity and surjectivity are equivalent to each other and to: nonzero determinant; invertible matrix/transformation; zero not an eigenvalue; transformation is a change of basis; linearly independent rows in matrix.
Matrix computations: determinant (characteristic polynomial), solving systems of linear equations (finding inverse matrix).
Eigenvalues, eigenvectors, eigenspaces; geometric/algebraic multiplicity; diagonalization.
Linear independence of eigenvectors for distinct eigenvalues.
Change of basis to ease finding matrix or for diagonalization.

III. Inner Products

Generalization of dot product, axiomatically defined; produce length and angle definitions.

Have triangle and Schwartz inequalities; scalars pull out as absolute value.
Orthogonal sets (linearly independent), orthonormal sets.
Computations: Fourier coefficients and the Gram-Schmidt process.
Isometric embedding; isometry (special kind of isomorphism).
Riesz representation theorem: inner products entirely cover linear functionals.
Orthogonality to a space, orthogonal complement.