Final Exam Practice Problems Volume 2 – Answers
Math 24 Winter 2012

(1) Find the kernel of the transformation given by matrix A below. What does that tell you about the transformation given by matrix B?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -3 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

Note that $A = B - 2I$. Row reduction shows the kernel of A is nontrivial, spanned by $(2, -1, 1)$, which indicates 2 is an eigenvalue for B with eigenspace spanned by $(2, -1, 1)$.

(2) Show that if A, B are diagonal $n \times n$ matrices, then $AB = BA$.

The product of two diagonal matrices is simply the product of the corresponding diagonal entries, which means commutativity of the matrix product comes directly from commutativity of products in $\mathbb{R}$.

(3) The *trace* of a square matrix A, $\operatorname{tr} A$, is the sum of A's diagonal entries.

(a) Find $\operatorname{tr} A$ for $A = \begin{bmatrix} 3 & 5 & -1 \\ 3 & -8 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

$3 - 8 + 2 = -3$

(b) It can be shown that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ for any two $n \times n$ matrices F and G. Using that fact, show that if A and B are similar, then $\operatorname{tr} A = \operatorname{tr} B$.

See problem 8 for work on similarity; $\operatorname{tr} A = \operatorname{tr}(Q^{-1}BQ) = \operatorname{tr}(Q^{-1}(BQ)) = \operatorname{tr}((BQ)Q^{-1}) = \operatorname{tr}(BI) = \operatorname{tr} B$.

(c) Suppose A is a diagonalizable $n \times n$ matrix. Show the trace of A is the sum of the eigenvalues of A (including multiplicity). [This is in fact true even if A is not diagonalizable, but don't worry about the general case.]

For A is diagonalizable, A must be similar to the diagonal matrix with diagonal entries the eigenvalues of A with multiplicity. The trace of that matrix is clearly the sum of the eigenvalues of A with multiplicity, and by part (b) it equals the trace of A.

(4) Let A, B be $n \times n$ matrices with rank k and ℓ, respectively. Put an upper bound on the rank of AB.

The rank of a matrix is the dimension of its image. The image of A has dimension k, so certainly AB must have rank bounded above by k. However, the dimension of the image of a transformation is also bounded by the dimension of the domain, and since the image of B has dimension ℓ our effective domain is dimension ℓ. The rank of AB is bounded above by $\min\{k, \ell\}$.

(5) For a polynomial $f(x)$ in $\mathcal{P}(\mathbb{R})$, let $F(x)$ be the polynomial with constant term 0 such that $F'(x) = f(x)$. Is the map from $\mathcal{P}(\mathbb{R})$ to itself that takes each $f(x)$ to the corresponding $F(x)$ a linear transformation?

Certainly the antiderivative (with constant 0) of a finite degree polynomial is a finite degree polynomial. Since integration has the correct relationship to addition and scalar multiplication, and we have specified a zero constant of integration, the transformation is in fact linear. We do not need to separately check that it maps the zero vector to itself because that is a consequence of linearity.

(6) Show that if A and B are square and AB is invertible, then both A and B are invertible.

This is shown most simply by observing $\det(AB) = \det(A)\det(B)$, and if the left hand side is nonzero both factors of the right hand side must also be nonzero.

(7) Determine whether the following two linear transformations are invertible, and if so find the inverse.

(a) $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by $T(f(x)) = (f(0), f(1), f(-1))$.

We see if we can reconstruct a unique polynomial from any vector of $\mathbb{R}^3$. For $f(x) = a_0 + a_1x + a_2x^2$, $f(0) = a_0$, $f(1) = a_0 + a_1 + a_2$, and $f(-1) = a_0 - a_1 + a_2$. From (b_0, b_1, b_2) we clearly need $a_0 = b_0$, and then $a_1 + a_2 = b_1 - b_0$, $-a_1 + a_2 = b_2 - b_0$. The matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is invertible, so this has a solution for any choice of b_0, b_1, b_2, and T is invertible.

(b) $T : \mathcal{P}_2(\mathbb{R}) \to \mathrm{Mat}_{2,2}$ given by $T(f(x)) = \begin{bmatrix} f(0) & f(1) \\ f'(0) & f'(1) \end{bmatrix}$.

This is a transformation from a 3-dimensional space to a 4-dimensional space, and so is not invertible.

(8) Matrix A is *similar* to matrix B if there is an invertible matrix P such that $P^{-1}AP = B$.

(a) Show that if A is similar to B, B is similar to A.

If $P^{-1}AP = B$, then $A = PBP^{-1}$, so the invertible matrix P^{-1} shows B is similar to A.

(b) Find all matrices X such that I_n is similar to X.

We must have $X = P^{-1}I_nP$, but that simplifies to $X = I_n$.

(c) Suppose $A = QR$ where Q is invertible. Show A is similar to RQ.

We use Q for the similarity: $Q^{-1}AQ = Q^{-1}QRQ = RQ$.

(9) If W is a subspace of a vector space V and v is a vector in V, define $v + W = \{v + w : w \in W\}$ (a sub*set* of V).

(a) If $V = \mathbb{R}^2$ and $W = \mathcal{L}\{(1,1)\}$, geometrically describe all possible sets $v + W$.

W itself is the line $x = y$, and the sets $v + W$ are all lines with the same slope as $x = y$ but varying intercepts.

(b) Show that if $v \in W$, then $v + W = W$. [From here on, V is an arbitrary vector space.]

Since W is closed under vector addition, it is clear from the definition that $v + W \subseteq W$. We get equality because $-v \in W$; for any $w \in W$ we may add $-v + w$ to v to obtain w.

(c) Show that if $v \notin W$, then $(v + W) \cap W = \emptyset$.

Suppose the intersection is nonempty, so there are some $w1, w_2 \in W$ such that $v + w_1 = w_2$. Then $v = w_2 - w_1$, so since W is closed, $v \in W$.

(d) For what vectors v is $v + W$ a subspace of V?

For exactly the vectors in W. In the preceding parts we have seen $v \in W$ implies $v + W = W$, a subspace, and $v \notin W$ implies $v + W \cap W = \emptyset$, so $v + W$ does not contain the zero vector and hence cannot be a subspace.

(e) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

If $v_1 + W = v_2 + W$, then for every $w_1 \in W$ there is some $w_2 \in W$ such that $v_1 + w_1 = v_2 + w_2$, and vice-versa. But then $v_1 - v_2 = w_2 - w_1 \in W$. Conversely, if $v_1 - v_2 \in W$ and we take any $w_1 \in W$, $v_1 + w_1 = v_2 + (v_1 - v_2) + w_1$, which is the sum of v_2 with a vector of W. This shows $v_1 + W \subseteq v_2 + W$, and a similar argument shows the reverse inclusion, giving equality.

(f) Addition and scalar multiplication may be defined as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \quad \text{and} \quad a(v + W) = (av) + W.$$

Prove that these operations are well-defined; that is, show that if $v_1 + W = v_1' + W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W) \quad \text{and} \quad a(v_1 + W) = a(v_1' + W).$$

By part (e) we know $v_1 - v_1'$ and $v_2 - v_2'$ are elements of W, so $(v_1' + v_2') + w_1 = (v_1' + v_2') + (v_1 - v_1') + (v_2 - v_2') + w_2 = (v_1 + v_2) + w_2$ Symmetrically, every element of $(v_1 + v_2) + W$ is also in $(v_1' + v_2') + W$. The argument for scalar multiplication is similar.

(g) Parts (e) and (f) show that we can put an addition and scalar multiplication on the *quotient space* V/W, where the elements of V/W are the sets $v + W$ (each one may have multiple representations $v_1 + W, v_2 + W$, etc., but we have shown it does not alter the result of addition and multiplication to choose a different representation).
In fact, V/W is a vector space. What is its additive identity (zero vector)?

The additive identity of V/W is W: $W = \mathbf{0} + W$, so by part (f), $W + (v + W) = (\mathbf{0} + v) + W = v + W$.

(10) Suppose that with respect to the basis $\{(1,0,1),(0,1,0),(1,0,-1)\}$ the transformation T has the following matrix.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) What is the matrix for T with respect to the standard basis?

After finding the change of basis matrix that takes vectors to coordinates, we find the standard basis matrix for T is the following product:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 5/2 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 5/2 \end{bmatrix}$$

(b) What is the hundredth power of the matrix from part (a)?

T is a linear transformation, and it maps vectors to the same vectors regardless of the basis we choose. Therefore the hundredth power of the matrix in (a) may be found by taking the hundredth power of the diagonal matrix before multiplying with the change of basis matrices (this is also seen by the fact that they are inverses to each other, so if we line up a hundred copies of that trio of matrices the matched pairs will cancel out):

$$\frac{1}{2}\begin{bmatrix} 3^{100} + 2^{100} & 0 & 3^{100} - 2^{100} \\ 0 & 2 & 0 \\ 3^{100} - 2^{100} & 0 & 3^{100} + 2^{100} \end{bmatrix}$$

(11) If $\boldsymbol{X}, \boldsymbol{Y}$ are eigenvectors for the linear transformation T, is $\boldsymbol{X} + \boldsymbol{Y}$ an eigenvector for T?

Only if $\boldsymbol{X}, \boldsymbol{Y}$ are eigenvectors for the same eigenvalue and hence in the same eigenspace. Otherwise no, because $\boldsymbol{X} + \boldsymbol{Y}$ would have to be an eigenvector for yet a third eigenvalue (otherwise we would contradict the distinctness of the individual vectors' eigenvalues), but then we would contradict the linear independence of eigenvectors drawn from distinct eigenspaces.

(12) Consider the linear transformation $T : \mathbb{R}^5 \to \mathcal{P}_2(\mathbb{R})$ given by $T(a_1, a_2, a_3, a_4, a_5) = (a_1 + a_2)x^2 - (a_4 + a_5)x + a_2 - a_3$.

(a) What are the image and kernel of T?

To be in the kernel, a vector must have $a_1 + a_2 = a_4 + a_5 = a_2 - a_3 = 0$. That corresponds to a matrix with the following reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

This is a space of dimension 2, spanned by the set $\{(-1,1,1,0,0),(0,0,0,1,-1)\}$. Since the dimension formula tells us the dimension of the image will be $\dim(\mathbb{R}^5) - \dim(\ker(T)) = 5-2 = 3 = \dim(\mathcal{P}_2(\mathbb{R}))$, the image is all of $\mathcal{P}_2(\mathbb{R})$.

(b) Find an orthonormal basis for the kernel of T, with respect to the standard scalar product on $\mathbb{R}^5$.

The basis is already orthogonal, so we need only divide each vector by its length. We get $\{(-1/\sqrt{3},1/\sqrt{3},1/\sqrt{3},0,0),(0,0,0,1/\sqrt{2},-1/\sqrt{2})\}$.