Final Exam Practice Problems Volume 2

Math 24 Winter 2012

(1) Find the kernel of the transformation given by matrix A below. What does that tell you about the transformation given by matrix B?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -3 \\ 0 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & -3 \\ 0 & 1 & 3 \end{bmatrix}$$

(2) Show that if A, B are diagonal $n \times n$ matrices, then $AB = BA$.

(3) The *trace* of a square matrix A, $\operatorname{tr} A$, is the sum of A's diagonal entries.

(a) Find $\operatorname{tr} A$ for $A = \begin{bmatrix} 3 & 5 & -1 \\ 3 & -8 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

(b) It can be shown that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$ for any two $n \times n$ matrices F and G. Using that fact, show that if A and B are similar, then $\operatorname{tr} A = \operatorname{tr} B$.

(c) Suppose A is a diagonalizable $n \times n$ matrix. Show the trace of A is the sum of the eigenvalues of A (including multiplicity). [This is in fact true even if A is not diagonalizable, but don't worry about the general case.]

(4) Let A, B be $n \times n$ matrices with rank k and ℓ, respectively. Put an upper bound on the rank of AB.

(5) For a polynomial $f(x)$ in $\mathcal{P}(\mathbb{R})$, let $F(x)$ be the polynomial with constant term 0 such that $F'(x) = f(x)$. Is the map from $\mathcal{P}(\mathbb{R})$ to itself that takes each $f(x)$ to the corresponding $F(x)$ a linear transformation?

(6) Show that if A and B are square and AB is invertible, then both A and B are invertible.

(7) Determine whether the following two linear transformations are invertible, and if so find the inverse.

(a) $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ given by $T(f(x)) = (f(0), f(1), f(-1))$.

(b) $T : \mathcal{P}_2(\mathbb{R}) \to \mathrm{Mat}_{2,2}$ given by $T(f(x)) = \begin{bmatrix} f(0) & f(1) \\ f'(0) & f'(1) \end{bmatrix}$.

(8) Matrix A is *similar* to matrix B if there is an invertible matrix P such that $P^{-1}AP = B$.

(a) Show that if A is similar to B, B is similar to A.

(b) Find all matrices X such that I_n is similar to X.

(c) Suppose $A = QR$ where Q is invertible. Show A is similar to RQ.

(9) If W is a subspace of a vector space V and v is a vector in V, define $v+W = \{v+w : w \in W\}$ (a sub*set* of V).

(a) If $V = \mathbb{R}^2$ and $W = \mathcal{L}\{(1,1)\}$, geometrically describe all possible sets $v + W$.

(b) Show that if $v \in W$, then $v + W = W$. [From here on, V is an arbitrary vector space.]

(c) Show that if $v \notin W$, then $(v + W) \cap W = \emptyset$.

(d) For what vectors v is $v + W$ a subspace of V?

(e) Prove that $v_1 + W = v_2 + W$ if and only if $v_1 - v_2 \in W$.

(f) Addition and scalar multiplication may be defined as follows:

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \quad \text{and} \quad a(v + W) = (av) + W.$$

Prove that these operations are well-defined; that is, show that if $v_1+W = v_1'+W$ and $v_2 + W = v_2' + W$, then

$$(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W) \quad \text{and} \quad a(v_1 + W) = a(v_1' + W).$$

(g) Parts (e) and (f) show that we can put an addition and scalar multiplication on the *quotient space* V/W, where the elements of V/W are the sets $v + W$ (each one may have multiple representations $v_1 + W, v_2 + W$, etc., but we have shown it does not alter the result of addition and multiplication to choose a different representation).
In fact, V/W is a vector space. What is its additive identity (zero vector)?

(10) Suppose that with respect to the basis $\{(1,0,1),(0,1,0),(1,0,-1)\}$ the transformation T has the following matrix.

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) What is the matrix for T with respect to the standard basis?

(b) What is the hundredth power of the matrix from part (a)?

(11) If $\boldsymbol{X}, \boldsymbol{Y}$ are eigenvectors for the linear transformation T, is $\boldsymbol{X} + \boldsymbol{Y}$ an eigenvector for T?

(12) Consider the linear transformation $T : \mathbb{R}^5 \to \mathcal{P}_2(\mathbb{R})$ given by $T(a_1, a_2, a_3, a_4, a_5) = (a_1 + a_2)x^2 - (a_4 + a_5)x + a_2 - a_3$.

(a) What are the image and kernel of T?

(b) Find an orthonormal basis for the kernel of T, with respect to the standard scalar product on $\mathbb{R}^5$.