Final Exam Practice Problems
Math 24 Winter 2012

(1) The Jordan product of two $n \times n$ matrices is defined as $A \otimes B = \frac{1}{2}(AB + BA)$, where the products inside the parentheses are standard matrix product. Is the set of all $n \times n$ matrices, with standard scalar multiplication and vector addition defined as a Jordan product, a vector space?

(2) Let $\boldsymbol{V}_1 = (1, 1, 1, 1)$, $\boldsymbol{V}_2 = (1, -1, 1, -1)$, and $\boldsymbol{V}_3 = (1, 1, -1, -1)$.
   (a) Show that $\{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3\}$ is an orthogonal set.
   (b) Find a $\boldsymbol{V}_4$ so that $\{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \boldsymbol{V}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.
   (c) Turn your basis from (b) into an orthonormal basis.

(3) For each matrix below: (a) Find the inverse or show it does not exist; (b) find the characteristic polynomial, all eigenvalues, and their associated eigenspaces, and (c) diagonalize the matrix if possible, giving a basis relative to which it has that diagonal form.

$$\begin{bmatrix} 3 & 2 & -1 \\ -6 & -1 & -4 \\ -6 & 2 & -10 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & 1 \\ -3 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

Hint for third matrix: 5 is an eigenvalue. Though the first matrix has an unpleasant characteristic polynomial computation, it factors without too much difficulty.

(4) For each pair of matrix properties below, there are three possible relationships: If you have x you must have y and vice-versa, if you have x you must have y but you may have y without x, or you can never have x and y simultaneously. Determine which one holds between each pair. Does your answer change if you consider only matrices that are neither the zero matrix nor the identity matrix?
   (a) diagonal
   (b) symmetric
   (c) nilpotent
   (d) idempotent
   (e) invertible

(5) Determine conditions on h and k such that the following system of linear equations has (i) infinitely many solutions, (ii) a unique solution, and (iii) no solutions.

$$\begin{aligned} x + 3y &= k \\ 4x + hy &= 8 \end{aligned}$$

(6) Assuming A, B, C, X are all $n \times n$ matrices and the first three are invertible, solve $AX + B = CA$ for X.

(7) Suppose the 2×2 matrix A has eigenvalue 2, with eigenvector $(1, 1)$, and eigenvalue -5, with eigenvector $(-1, 1)$. Use change of basis to find A.

(8) The augmented matrix of a system of three linear equations in three unknowns has been row reduced to the following form. What are the solutions to the system, if any?

$$\left[\begin{array}{ccc|c} 1 & 5 & 2 & -1 \\ 0 & 2 & -4 & 8 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

(9) Let $P : V \to V$ be a projection, and let $\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k\}$ be a basis for $\operatorname{Im} P$. Suppose that this basis is extended to a basis for all of V, $\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k, \boldsymbol{B}_1, \ldots, \boldsymbol{B}_\ell\}$.

(a) For each i, $1 \leq i \leq \ell$, let $\boldsymbol{C}_i = \boldsymbol{B}_i - P(\boldsymbol{B}_i)$. Show that $\{\boldsymbol{A}_1, \ldots, \boldsymbol{A}_k, \boldsymbol{C}_1, \ldots, \boldsymbol{C}_\ell\}$ is a basis for V.

(b) Find the matrix for P with respect to the basis in part (a).

(10) How many isometries are there from $\mathbb{R}$ to $\mathbb{R}$, with the standard inner product?

(11) In a three-dimensional vector space V, two bases are $\mathcal{M} = \{\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3\}$ and $\mathcal{N} = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$. Given the following relationships between $\mathcal{M}$ and $\mathcal{N}$, find the change-of-basis matrices from $\mathcal{M}$ to $\mathcal{N}$ and from $\mathcal{N}$ to $\mathcal{M}$.

$$\boldsymbol{m}_1 = 2\boldsymbol{n}_2 + \boldsymbol{n}_3$$
$$\boldsymbol{m}_2 = -\boldsymbol{n}_1$$
$$\boldsymbol{m}_3 = \boldsymbol{n}_1 - \boldsymbol{n}_3$$

(12) Let V be a vector space and $S \subseteq V$ be a spanning set for V. Suppose $A \subseteq S$ is linearly independent, but $A \cup \{x\}$ is linearly dependent for all $x \in S - A$. Prove A is a basis for V.

(13) Determine whether the set $\{x^2 + 2x - 1,\ 3x + 2,\ -x^2 - x + 3\}$ is a basis for $\mathcal{P}_2(\mathbb{R})$.

(14) Prove that if E_1 and E_2 are eigenspaces for $T : V \to V$, then either $E_1 = E_2$ or $E_1 \cap E_2 = \{\boldsymbol{0}\}$.