Math 24
Winter 2010
Special Assignment due Monday, March 1
Sample Solution

Let V be any vector space over F and W be a subspace of V. We know that V/W is a vector space, and that $T(x) = x + W$ is a linear transformation from V to V/W.

Assignment: Let α be a basis for W that can be extended to a basis $\alpha \cup \beta$ for V (where $\alpha \cap \beta = \emptyset$). Show that $\{x + W \mid x \in \beta\}$ is a basis for V/W.

Note that we have not assumed V is finite-dimensional, so α and β may be infinite.

On the next pages are two solutions to this problem. The first proof is specific to the situation here.

The second proof actually shows a more general result: Whenever $T : V \to Z$ is a linear transformation, $W = N(T)$, and α is a basis for W that can be extended to a basis $\alpha \cup \beta$ for V (where $\alpha \cap \beta = \emptyset$), we have that $\{T(x) \mid x \in \beta\}$ is a basis for $R(T)$. This proves our result, since we know $R(T) = V/W$.

If V is finite-dimensional, this more general result proves the Dimension Theorem, since $dim(W) = size(\alpha)$, $dim(V) = size(\alpha) + size(\beta)$, and (by this result) $dim(R(T)) = size(\beta)$. This is actually how the Dimension Theorem is proven.

Solution I:

Let $B = \{x + W \mid x \in \beta\}$. To show that B is a basis for V/W, we must show that B spans V/W and that B is linearly independent.

First, we show that B spans V/W. Let $x + W$ be any element of V/W. We must show that $x + W$ is in the span of B.

Because $\alpha \cup \beta$ spans V, we may write

$$x = a_1 w_1 + \cdots + a_m w_m + b_1 v_1 + \cdots + b_n v_n,$$

where $w_i \in \alpha$ and $v_i \in \beta$. Set $y = b_1 v_1 + \cdots + b_n v_n$. Then $x - y = a_1 w_1 + \cdots + a_m w_m \in W$, so by an earlier assignment,

$$x + W = y + W = (b_1 v_1 + \cdots + b_n v_n) + W = b_1(v_1 + W) + \cdots + b_n(v_n + W).$$

We have expressed $x + W$ as a linear combination of elements of B, which shows that $x + W$ is in the span of B.

Now we show that B is linearly independent. To do this, we suppose that some linear combination of elements of B equals zero,

$$b_1(v_1 + W) + \cdots + b_n(v_n + W) = 0_{V/W},$$

where the v_i are distinct elements of β. We must show that $b_1 = \cdots = b_n = 0$.

We have that

$$0_{V/W} = b_1(v_1 + W) + \cdots + b_n(v_n + W) = (b_1 v_1 + \cdots + b_n v_n) + W.$$

Now $0_{V/W} = 0 + W = W$. Since we showed that $x \in x + W$, we can conclude that

$$b_1 v_1 + \cdots + b_n v_n \in W.$$

Since α is a basis for W, we can write

$$b_1 v_1 + \cdots + b_n v_n = a_1 w_1 + \cdots + a_m w_m,$$

where the w_i are distinct elements of α. Since $\alpha \cap \beta = \emptyset$, the v_i and w_i are distinct elements of $\alpha \cup \beta$, and we have

$$b_1 v_1 + \cdots + b_n v_n - a_1 w_1 - \cdots - a_m w_m = 0.$$

Since $\alpha \cup \beta$ is linearly independent, we must have $b_1 = \cdots = b_n = a_1 = \cdots = a_n = 0$.

Solution II:

We will use the fact that $T(x) = x + W$ is a linear transformation from V to V/W with $R(T) = V/W$ and $N(T) = W$. We let $B = \{x + W \mid x \in \beta\} = \{T(x) \mid x \in \beta\}$, and show B is a basis for $R(T)$. To do this, we must show that B spans $R(T)$ and that B is linearly independent.

First, we show that B spans $R(T)$. Let $T(x)$ be any element of $R(T)$. We must show that $T(x)$ is in the span of B.

Because $\alpha \cup \beta$ spans V, we may write

$$x = a_1 w_1 + \cdots + a_m w_m + b_1 v_1 + \cdots + b_n v_n,$$

where $w_i \in \alpha$ and $v_i \in \beta$. Since $w_i \in \alpha \subseteq W = N(T)$, we know $T(w_i) = 0$. Now

$$T(x) = T(a_1 w_1 + \cdots + a_m w_m + b_1 v_1 + \cdots + b_n v_n) =$$

$$a_1 T(w_1) + \cdots + a_m T(w_m) + b_1 T(v_1) + \cdots + b_n T(v_n) =$$

$$a_1(0) + \cdots + a_m(0) + b_1 T(v_1) + \cdots + b_n T(v_n) = b_1 T(v_1) + \cdots + b_n T(v_n).$$

We have expressed $T(x)$ as a linear combination of elements of B, which shows that $T(x)$ is in the span of B.

Now we show that B is linearly independent. To do this, we suppose that some linear combination of elements of B equals zero,

$$b_1 T(v_1) + \cdots + b_n T(v_n) = 0,$$

where the v_i are distinct elements of β. We must show that $b_1 = \cdots = b_n = 0$.

We have

$$0 = b_1 T(v_1) + \cdots + b_n T(v_n) = T(b_1 v_1 + \cdots + b_n v_n).$$

This means that

$$b_1 v_1 + \cdots + b_n v_n \in N(T) = W.$$

Since α is a basis for W, we can write

$$b_1 v_1 + \cdots + b_n v_n = a_1 w_1 + \cdots + a_m w_m,$$

where the w_i are distinct elements of α. Since $\alpha \cap \beta = \emptyset$, the v_i and w_i are distinct elements of $\alpha \cup \beta$, and we have

$$b_1 v_1 + \cdots + b_n v_n - a_1 w_1 - \cdots - a_m w_m = 0.$$

Since $\alpha \cup \beta$ is linearly independent, we must have $b_1 = \cdots = b_n = a_1 = \cdots = a_n = 0$.