Math 24
Winter 2010
Comment on Special Assignment due Monday, February 22

We define a function T from V to V/W by $T(x) = x + W$.

Many people are confused about how to determine $N(T)$. Many people are also mixing up elements of V and elements of V/W. Vectors in V and cosets in V/W are entirely different animals.

Here is an example:

Suppose $V = \mathbb{R}^2$ and W is the x-axis. Then

$$(a,b) + W = \{(a,b) + (x,y) \mid (x,y) \in W\} = \{(a,b) + (x,0) \mid x \in \mathbb{R}\} = \{(x+a,b) \mid x \in \mathbb{R}\}.$$

That is, $(a,b) + W$ is the line $y = b$. Notice that (a,b) is on that line.

So V/W is the set of horizontal lines in $\mathbb{R}^2$, and for $v \in \mathbb{R}^2$, $v + W$ is the horizontal line containing v.

To add cosets (in this case, lines) and multiply them by scalars, we use the definitions of these operations. The line $y = b$ is the coset $(0,b) + W$. (It is $(a,b) + W$ for any a. We just choose $a = 0$ to make things simple.) Now if X is the line $y = b$ and Y is the line $y = c$, we can write $X = (0,b) + W$ and $Y = (0,c) + W$, so

$$X + Y = ((0,b) + W) + ((0,c) + W) = ((0,b) + (0,c)) + W = (0, b+c) + W;$$

$$rX = r((0,b) + W) = (r(0,b) + W) = (0, rb) + W.$$

That is,

$$(\text{line } y = b) + (\text{line } y = c) = (\text{line } y = (b+c));$$

$$r(\text{line } y = b) = (\text{line } y = rb).$$

Recall that when you showed V/W satisfies the first four vector space axioms, you showed that the additive identity of V/W is

$$0_{V/W} = 0 + W = \{0 + w \mid w \in W\} = \{w \mid w \in W\} = W,$$

so in our example, the zero element of V/W is the x-axis.

Therefore in this example,

$$N(T) = \{v \mid T(v) = 0\} = \{v \mid v + W = x - axis\} = \{v \mid v \text{ is on the } x - axis\} = x - axis.$$

Also, in this example, any element of V/W, any horizontal line X, is the horizontal line containing v for some point v, so $X = v + W = T(v)$, which shows X is in the range of T. We just showed that every element of V/W is in $R(T)$, so T is onto, and $R(T) = V/W$.