Math 24

Winter 2010

Special Assignment due Monday, February 1

Let V be any vector space and W be a subspace of V. For any vector x in V, we define the *coset* of W containing x to be

$$x + W = \{x + w \mid w \in W\}.$$

We denote the collection of cosets of W in V by V/W.

For your last assignment, you proved that addition of cosets defined by

$$(x + W) + (y + W) = (x + y) + W$$

is well-defined.

Assignment: Prove that V/W, with addition defined as above, satisfies the first four vector space axioms.

Note: We can make a similar definition for other sorts of structures and substructures. For example, the integers $\mathbb{Z}$ with addition and multiplication form a "commutative ring with unity." This is a structure that satisfies all the axioms for a field except possibly the existence of multiplicative inverses. The set of multiplies of n

$$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$$

is a kind of substructure of $\mathbb{Z}$ called an "ideal." This means it is closed under addition, and also under multiplication by any element of $\mathbb{Z}$. Now if we define cosets of $n\mathbb{Z}$ the same way we did above,

$$x + n\mathbb{Z} = \{x + m \mid m \in n\mathbb{Z}\},$$

we can define addition and multiplication of cosets

$$(x + n\mathbb{Z}) + (y + n\mathbb{Z}) = (x + y) + n\mathbb{Z} \quad \text{and} \quad (x + n\mathbb{Z})(y + n\mathbb{Z}) = (xy) + n\mathbb{Z}.$$

We get the structure $\mathbb{Z}/n\mathbb{Z}$, whose elements are cosets $0 + n\mathbb{Z}$, $1 + n\mathbb{Z}$, $\ldots (n-1) + n\mathbb{Z}$. We called these elements $\overline{0}, \overline{1}, \ldots \overline{n-1}$.

That is where the notation $\mathbb{Z}/n\mathbb{Z}$ comes from.