Math 24
Winter 2010
Quiz 7

1. (a) You can find a basis for the subspace of $\mathbb{R}^4$ spanned by $\{(3,0,4,1),(1,3,2,1),(5,6,8,3),(1,0,5,1)\}$ if you know the reduced row echelon form of a certain matrix A.

   i. What is A?

   $$A = \begin{pmatrix} 3 & 1 & 5 & 1 \\ 0 & 3 & 6 & 0 \\ 4 & 2 & 8 & 5 \\ 1 & 1 & 3 & 1 \end{pmatrix}$$, or any other matrix with these columns.

   ii. If the reduced row echelon form of A is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, what is a basis for this subspace?

   $\{(3,0,4,1),(1,3,2,1),(1,0,5,1)\}$, or any three columns of A that correspond to linearly independent columns of the reduced row echelon form.

(b) You can check whether the set $\{(1,2,3),(1,0,1),(2,2,1)\} \subseteq \mathbb{R}^3$ is linearly independent by finding the determinant of a certain matrix B.

   i. What is B?

   $$B = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$, or any matrix with these three vectors as columns or as rows.

   ii. If the determinant of B is 6, is the set linearly independent?

   Yes, because $det(B) \neq 0$ means that B is invertible, so B has rank 3.

2. $T : \mathbb{R}^3 \to \mathbb{R}^3$ is determined by its action on the vectors of the basis $\beta = \{v_1, v_2, v_3\}$, as follows: $T(v_1) = 3v_1$, $T(v_2) = v_2 + v_1$, and $T(v_3) = v_3 - v_1$.

   (a) What is $[T]_\beta$?

   The columns of $[T]_\beta$ are the β coordinates of $T(v_1)$, $T(v_2)$, and $T(v_3)$.
   $T(v_1) = 3v_1 = 3(v_1) + 0(v_2) + 0(v_3)$,
   $T(v_) = v_2 + v_1 = 1(v_1) + 1(v_2) + 0(v_3)$,
   $T(v_1) = v_3 - v_1 = (-1)(v_1) + 0(v_2) + 1(v_3)$, so
   $$[T]_\beta = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (b) If A is the matrix of T in the standard basis, what is $det(A)$? How do you know?

   $det(A) = 3$.
   A and B have the same determinant because they are similar matrices, and the determinant of B is the product of its diagonal entries, because B is upper triangular.

3. Find an eigenvalue, and the set of corresponding eigenvectors, of the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

The eigenvalues of the matrix are the roots of its characteristic polynomial, which is $det \begin{pmatrix} 1-t & 0 \\ 2 & 1-t \end{pmatrix} = (1-t)^2$. There is only one root, $t = 1$, so only one eigenvalue, $\lambda = 1$.

To find eigenvectors of A corresponding to eigenvalue λ, solve the system $Av = \lambda v$, or $(A - \lambda I)v = 0$. In this case we are solving

$$\begin{pmatrix} 1-1 & 0 \\ 2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ or}$$

$\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, whose solution set is $\left\{ s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \right\}$. The set of eigenvectors is $\left\{ s \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; s \in \mathbb{R} \;\&\; s \neq 0 \right\}$.