Math 24
Winter 2010
Quiz 6

1. Consider the following system of linear equations.

$$x_1 + x_2 + x_3 + x_4 + 4x_5 = 8$$
$$2x_1 + 2x_2 + 3x_3 + 1x_4 + 8x_5 = 16$$
$$-2x_1 - 2x_2 - 4x_5 = -8$$
$$x_1 + x_2 + 4x_3 + 4x_4 + 10x_5 = 20$$

The reduced row echelon form of the augmented matrix of the system is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Give the complete solution of the system.

The reduced row echelon form gives us the equivalent system

$$\begin{aligned} x_1 + x_2 + 2x_5 &= 4 \\ x_3 + x_5 &= 2 \\ x_4 + x_5 &= 2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_1 &= -x_2 - 2x_5 + 4 \\ x_3 &= -x_5 + 2 \\ x_4 &= -x_5 + 2 \end{aligned}$$

Therefore we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -x_2 - 2x_5 + 4 \\ x_2 \\ -x_5 + 2 \\ -x_5 + 2 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \\ 0 \end{pmatrix}.$$

2. Let A be the coefficient matrix of the system of linear equations given in question (1). That is, we have $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & 1 & 8 \\ -2 & -2 & 0 & 0 & -4 \\ 1 & 1 & 4 & 4 & 10 \end{pmatrix}$. Specify:

   (a) The null space of L_A is the solution space of the corresponding homogeneous system. This is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, as we can see from the solution to the given system, which is the complete solution to the homogeneous system plus one particular solution to the given system. Therefore we have:

A basis for the null space of L_A: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

(b) Elementary row operations do not change linear dependencies among columns. Because columns 1, 3, and 4 form a basis for the space spanned by the columns of the reduced row echelon form of A, the same is true for A. Therefore we have:

A basis for the range of L_A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 4 \end{pmatrix} \right\}$.

(c) The rank of A: 3.

3. Use elementary row operations to transform the matrix $A = \begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$ into upper triangular form. Find the determinant of A.

In order, interchange rows 1 and 2, add (-2) times row 1 to row 3, add (1) times row 2 to row 3. The steps of this transformation are:

$$\begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & -1 \\ 4 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & -3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The determinant of this upper triangular matrix is the product of its diagonal entries, or 12. Since we performed one interchange of rows, which changes the sign of the determinant, we see

$$det(A) = -12.$$

We can check this by computing $det(A)$ by cofactor expansion.