Math 24
Winter 2010
Thursday, January 21

(1.) TRUE or FALSE? In these exercises, V and W are finite-dimensional vector spaces over a field F, and T is a function from V to W.

a. If T is linear, then T preserves sums and scalar products.

b. If $T(x+y) = T(x) + T(y)$, then T is linear.

c. T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.

d. If T is linear, then $T(0_V) = 0_W$.

e. If T is linear, then $nullity(T) + rank(T) = dim(W)$.

f. If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.

g. If $T : V \to W$ and $U : V \to W$ are both linear and agree on a basis for V, then $T = U$.

h. Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

i. Recall that we can consider $\mathbb{R}$ to be a vector space over itself. Any function $T : \mathbb{R} \to \mathbb{R}$ of the form $T(x) = mx + b$, where m and b are constants in $\mathbb{R}$, is linear.

j. The words "range," "image," and "codomain" all mean the same thing.

k. If $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ is linear and $N(T)$ is the subspace of diagonal matrices in $M_{2\times 2}(\mathbb{R})$, then T is not onto.

(2.) Explain why we know that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ is not linear.

a. $T(a_1, a_2) = (1, a_2)$.

b. $T(a_1, a_2) = (a_1, (a_1)^2)$.

c. $T(a_1, a_2) = (\sin(a_1), 0)$.

d. $T(a_1, a_2) = (|a_1|, a_2)$.

e. $T(a_1, a_2) = (a_1 + 1, a_2)$.

(3.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x - y, 2y - z, 4x - z)$.

a. Find a basis for $N(T)$.

b. Find a basis for $R(T)$.

c. Find the nullity and rank of T. Verify the dimension theorem (in the case of T).

d. Is T one-to-one? How can you tell from the nullity and/or rank of T?

e. Is T onto? How can you tell from the nullity and/or rank of T?