Math 24
Winter 2010
Friday, January 15

(1.) TRUE or FALSE?

a. If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

b. Any set containing the zero vector is linearly dependent.

c. The empty set is linearly dependent.

d. Subsets of linearly dependent sets are linearly dependent.

e. Subsets of linearly independent sets are linearly independent.

f. If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

g. A set $\{x\}$ with one element is linearly dependent if and only if x is the zero vector.

h. A set $\{x, y\}$ with two elements is linearly dependent if and only if one of those elements is a scalar multiple of the other.

i. A set of four vectors in $\mathbb{R}^3$ must be linearly dependent.

j. The zero vector space has no basis.

k. Every vector space that is generated by a finite set has a finite basis.

l. Every vector space has a finite basis.

m. A vector space cannot have more than one basis.

n. If a vector space has a finite basis, then the number of vectors in every basis is the same.

o. The dimension of $P_n(F)$ is n.

p. The dimension of $M_{m\times n}(F)$ is $m + n$.

q. Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.

r. If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.

s. Every subspace of a finite-dimensional vector space is finite-dimensional.

t. If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.

u. If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if it spans V.

v. A vector space V cannot have both a finite basis and an infinite basis.

w. There are subspaces W_1, W_2, W_3, and W_4 of $\mathbb{R}^3$ such that $W_1 \subset W_2 \subset W_3 \subset W_4$. (Recall that $X \subset Y$ means that X is a subset of Y that does not contain all the elements of Y. Another way to say this is that S is a *proper* subset of Y.)

x. If S is a linearly independent subset of a finite-dimensional vector space V, then there is a basis for V that contains S.

y. If S is a subset of a finite-dimensional vector space V that generates V, then some subset of S is a basis for V.

z. If V is an n-dimensional vector space, and $S = \{v_1, v_2, \ldots, v_n\} \subset V$, then S is linearly independent if and only if S generates V.

(2.) Theorem 1.8 in the textbook is the following.

Let V be a vector space and $\beta = \{u_1, u_2, \ldots, u_\}$ be a subset of V. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

(a.) What does the word "unique" in this theorem mean?

(b.) In the proof, the authors begin, "Let β be a basis for $V \ldots$," and conclude that if $v \in V$, then v is uniquely expressible as a linear combination of vectors of β. Then they say, "The proof of the converse is an exercise."

What is the converse?

(c.) How would you go about proving the converse? Don't write out a proof, just answer these questions:

What would you assume?

What two things must you then prove?

Do you see an easy way to prove one or both of them?

(3.) Determine whether the set $\{(1,2,1),(1,4,-1),(3,8,2)\} \subseteq \mathbb{R}^3$ is linearly independent.

(4.) Determine whether the set $\{1+x, x+x^2, x^2+x^3, x^3+1\} \subseteq P_3(\mathbb{Q})$ is linearly independent.

(5.) Determine whether these sets are linearly dependent or linearly independent without solving a system of linear equations.

(a.) $\{(1,2,3),\ (.25,9,4),\ (-\pi, ln(2), \sqrt{7}), (1,1,1)\} \subset \mathbb{R}^3$.

(b.) $\{1+x+x^2,\ 3x-4x^2, 3+x, 4+x-x^2\} \subset P_2(\mathbb{Q})$.

(c.) $\{(1,1,0),\ (1,-1,0),\ (1,1,1)\} \subset \mathbb{R}^3$.

(d.) $\{3,\ x+4,\ x^2-1\} \subset P_2(\mathbb{R})$.

(e.) $\{3+x^2,\ 2-4x^2, 14+8x^2\} \subset P_2(\mathbb{R})$.

(f.) $\left\{ \begin{pmatrix} 2 & 4i \\ 3 & -2 \end{pmatrix}, \begin{pmatrix} -i & i+2 \\ 4 & i \end{pmatrix}, \begin{pmatrix} 2+2i & .5 \\ .25i & -2-2i \end{pmatrix}, \begin{pmatrix} 0 & i \\ \pi - .25i & 0 \end{pmatrix} \right\} \subset M_{2\times 2}(\mathbb{C})$.
(Hint: Consider the traces of these matrices.)

(6.) Let $S = \{u_1, u_2, \cdots, u_n\}$ be a finite set of nonzero vectors. Prove by induction on n that if no vector in S is in the span of the earlier listed vectors ($u_{k+1} \notin span(u_1, \ldots, u_k)$ for all k with $1 \leq k < n$), then S is linearly independent. Theorem 1.7 of the textbook might help with the inductive step.

(It is possible to prove this theorem without using induction, as well.)

(Note that the converse to this theorem is also true: Let $S = \{u_1, u_2, \cdots, u_n\}$ be a finite set of nonzero vectors. If S is linearly independent, then no vector in S is in the span of the earlier listed vectors.)

(7.) Apply problem (6) to show the following are linearly independent:

(a.) The columns of a square upper triangular matrix all of whose diagonal entries are nonzero. (An upper triangular matrix is one that has only zeroes below the diagonal.)

(b.) Any set of nonzero polynomials of different degree.

(8.) Let V be a vector space over a field F of characteristic not equal to two. (Recall that F has characteristic equal to two just in case $1 + 1 = 0$ in F.)

(a.) Let u and v be two distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v,\, u - v\}$ is linearly independent.

(b.) What happens if F has characteristic equal to two?