Math 24
Winter 2010
Friday, January 8

Here is a sample answer to one of the problems we did in class:

(2.) Show the subset of $\mathcal{F}(\mathbb{R},\mathbb{R})$ consisting of all functions f such that, for every x, we have $f(x+2\pi) = f(x)$, is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$.

Proposition: The subset of $\mathcal{F}(\mathbb{R},\mathbb{R})$ consisting of all functions f such that, for every x, we have $f(x+2\pi) = f(x)$, is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$.

Proof: Call this subset W. We can write

$$W = \{f \mid f \in \mathcal{F}(\mathbb{R},\mathbb{R}), \text{ and for all } x,\ f(x+2\pi) = f(x)\}.$$

To show that W is a subspace, we must show that W contains the zero vector of $\mathcal{F}(\mathbb{R},\mathbb{R})$ and is closed under addition and multiplication by scalars.

The zero vector of $\mathcal{F}(\mathbb{R},\mathbb{R})$ is the constant function c_0 whose value at every point is 0. Now, for every x,

$$c_0(x) = 0 \text{ and } c_0(x+2\pi) = 0, \text{ and therefore } c_0(x) = c_0(x+2\pi),$$

which shows that $c_0 \in W$.

Now, to show that W is closed under addition and scalar multiplication, suppose that f and g are in W. This means that for every x we have

$$f(x+2\pi) = f(x) \text{ and } g(x+2\pi) = g(x).$$

Therefore, using the definition of addition of functions, for every x we have

$$(f+g)(x+2\pi) = f(x+2\pi) + g(x+2\pi) = f(x) + g(x) = (f+g)(x),$$

and therefore $f + g \in W$. This shows W is closed under addition. Similarly, letting a be any real number, and using the definition of multiplication of functions by scalars, for every x we have

$$(af)(x+2\pi) = a(f(x+2\pi)) = a(f(x)) = (af)(x),$$

and therefore $af \in W$. This shows W is closed under scalar multiplication, and that completes the proof.