Math 24
Winter 2010
Friday, January 8

(1.) What is the smallest subspace of $\mathbb{R}^3$ containing the vectors $(1, 1, 2)$ and $(0, 1, 1)$? (Be specific. Describe this subspace using an equation or equations.)

(2.) Show the subset of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ consisting of all functions f such that, for every x, we have $f(x + 2\pi) = f(x)$, is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

(3.) Show the following are subspaces of $M_{2\times 2}(\mathbb{R})$ (with the same operations).

(a.) Matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ whose entries satisfy the equation $ax + by + cz + dw = 0$ where a, b, c and d are constants.

(b.) Matrices $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ whose entries satisfy the system of equations

$$a_1x + b_1y + c_1z + d_1w = 0$$

$$a_2x + b_2y + c_2z + d_2w = 0$$

$$a_3x + b_3y + c_3z + d_3w = 0.$$

Hint: Use part (a) and Theorem 1.4 on page 19 of the textbook.

(4.) If W_1 and W_2 are subsets of a vector space V, we define

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1 \ \& \ w_2 \in W_2\}.$$

(a.) If V is $\mathbb{R}^3$, W_1 is the x-axis, and W_2 is the z-axis, what is $W_1 + W_2$?

(b.) Show that if W_1 and W_2 are both subspaces of V, then so is $W_1 + W_2$.

(c.) Show that if W is any subspace of V such that $W_1 \subseteq W$ and $W_2 \subseteq W$, then $W_1 + W_2 \subseteq W$.

(d.) Notice that this means that if W_1 and W_2 are both subspaces of V, then $W_1 + W_2$ is the smallest subspace of V that contains both W_1 and W_2, as in the example in part (a).

If W_1 and W_2 are both subspaces of V and $W_1 \cap W_2 = \{0\}$, as in the example in part (a), it turns out that every element of $W_1 + W_2$ can be expressed *uniquely* as $w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$. In this case we say that $W_1 + W_2$ is the *direct sum* of W_1 and W_2, and denote this direct sum as $W_1 \oplus W_2$. (Note that this is not exactly the same as the definition of direct sum in an earlier exercise sheet. They are, however, essentially the same, which we can explain precisely once we have learned about vector space isomorphisms.)

That's right, there is nothing to do for part (d), other than make sure you understand the first sentence.

(5.) If W is a subspace of V, and x is a vector in V, the set

$$\{x\} + W = \{x + w \mid w \in W\}$$

is also denoted $x + W$ and is called the *coset* of W containing x.

(a.) If W is the line in $\mathbb{R}^2$ with equation $x - y = 0$, give a geometric description of the coset $(0, 1) + W$. (Be specific. If the coset were a circle — which it isn't — you should not just say "a circle," you should give the center and radius of the circle.)

(b.) Give a geometric description of all the cosets of the subspace W of $\mathbb{R}^2$ in part (a).

(c.) Show that if W is a subspace of a vector space V, and x and y are any elements of V, then:

(i.) If $x - y \in W$ then $x + W = y + W$. (To show two sets are equal, show any element of one must also be in the other, and vice versa.)

(ii.) If $x - y \notin W$ then $(x + W) \cap (y + W) = \emptyset$.

(iii.) $x \in x + W$.