Math 24
Winter 2010
Friday, February 26

For this problem, $V = \mathbb{R}^n$, and W is an m-dimensional subset of V. We define

$$W^\perp = \{v \mid w \cdot v = 0 \text{ for all } w \in W\},$$

where $\cdot$ denotes the familiar dot product.

For example, if $n = 3$ and $m = 2$, then the subspace W is a plane through the origin, and $W^\perp$ is the line through the origin perpendicular to that plane. If $n = 3$ and $m = 1$, then the subspace W is a line through the origin, and $W^\perp$ is the plane through the origin perpendicular to that line.

(1.) Show that $W^\perp$ is a subspace of V.

We must show that $0 \in W^\perp$ and that $W^\perp$ is closed under addition and multiplication by scalars. To do this, we recall from multivariable calculus that if w is any fixed element of $\mathbb{R}^n$, then for any v_1 and v_2 in $\mathbb{R}^n$ and any real number r, we have the following properties.

$$w \cdot \vec{0} = 0;$$

$$w \cdot (v_1 + v_2) = w \cdot v_1 + w \cdot v_2;$$
$$w \cdot rv_1 = r(w \cdot v_1).$$

To show $0 \in W^\perp$, suppose $w \in W$. Then $w \cdot 0 = 0$ by the first property. This shows $0 \in W^\perp$.

To show $W^\perp$ is closed under addition, suppose $v_1 \in W^\perp$ and $v_2 \in W^\perp$. Then we must show that $v_1 + v_w \in W^\perp$. That is, we must show that $w \cdot (v_1 + v_2) = 0$ for all $w \in W$.

To do this, suppose $w \in W$. By the second property, $w \cdot (v_1 + v_2) = w \cdot v_1 + w \cdot v_2$. Since v_1 and v_w are in $W^\perp$, we have $w \cdot v_1 = 0$ and $w \cdot v_2 = 0$. Therefore, $w \cdot (v_1 + v_2) = w \cdot v_1 + w \cdot v_2 = 0 + 0 = 0$, which is what we needed to show.

The proof that $W^\perp$ is closed under multiplication by scalars is similar, using the third property.

(2.) Suppose that $\beta = \{w_1, w_2, \ldots, w_m\}$ is a basis for W. Show that for any $v \in V$ we have

$$v \in W^\perp \iff w_i \cdot v = 0 \text{ for } i = 1, 2, \ldots, m.$$

The $\Longrightarrow$ direction is immediate, because if $w \cdot v = 0$ for every $w \in W$, then surely $w \cdot v = 0$ if w is one of the basis vectors of β.

To show the $\Longleftarrow$ direction, suppose that $w_i \cdot v = 0$ for $i = 1, 2, \ldots, m$ and choose $w \in W$. We must show $w \cdot v = 0$.

Because β is a basis, we can write $w = a_1 w_1 + a_2 w_2 + \cdots + a_m w_m$. Now, repeatedly applying the second and third properties above, we have

$$w \cdot v = (a_1 w_1 + a_2 w_2 + \cdots + a_m w_m) \cdot v = a_1(w_1 \cdot v) + a_2(w_2 \cdot v) + \cdots + a_m(w_m \cdot v) =$$

$$a_1(0) + a_2(0) + \cdots + a_m(0) = 0.$$

(3.) Show that the dimension of $W^\perp$ is $n - m$.

If we write the vectors w_1, w_2, $\ldots$, w_m as row vectors, and let A be the $m \times n$ matrix with those vectors as rows, we have

$$Av = \begin{pmatrix} w_1 \cdot v \\ w_2 \cdot v \\ \vdots \\ w_m \cdot v \end{pmatrix}.$$

Now by problem (2), we have $v \in W^\perp$ if and only if $Av = 0$, so $W^\perp$ is the null space of L_A.

Because the rows of A are linearly independent (they are elements of a basis for W), the rank of A is m. The rank of A is the dimension of $R(L_A)$. Since the domain of L_A is $\mathbb{R}^n$, by the Dimension Theorem, the null space of L_A has dimension $n - m$. So $dim(W^\perp) = n - m$.

(4.) Show that $V = W \oplus W^\perp$.

First we show $W \cap W^\perp = \{0\}$. Suppose $w \in W$ and $w \in W^\perp$; we must show $w = 0$. Because $w \in W^\perp$, the dot product of w with any element of W is 0. But w itself is an element of W, so $w \cdot w = 0$. This means $w = 0$.

The sum $W + W^\perp$ is some subspace of $\mathbb{R}^n$; we'll call it Z. Because $W \cap W^\perp = \{0\}$, we have $Z = W \oplus W^\perp$. By one of the characterizations of direct sum, the union of a basis for W (which has m elements) and a basis for $W^\perp$ (which has $n - m$ elements) is a basis for Z.

But now the dimension of Z is $m + (n - m) = n$, and so Z must be all of $\mathbb{R}^n$. Hence

$$V = \mathbb{R}^n = Z = W \oplus W^\perp.$$