Math 24
Winter 2010
Wednesday, February 24

(1.) TRUE or FALSE?

(a.) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

FALSE. There are linear operators with no eigenvalues, and problem (1) from last time gives another counterexample. However, there are *at most* n distinct eigenvalues.

(b.) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

TRUE. Any nonzero scalar multiple of an eigenvector is also an eigenvector.

(c.) There exists a square matrix with no eigenvectors.

TRUE. The matrix of any rotation of $\mathbb{R}^2$ through an angle that is not an integer multiple of π is an example.

(d.) Eigenvalues must be nonzero scalars.

FALSE. Zero can be an eigenvalue.

(e.) Any two eigenvectors are linearly independent.

FALSE. Any nonzero scalar multiple of an eigenvector is also an eigenvector. However, two eigenvectors corresponding to different eigenvalues must be linearly independent.

(f.) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

FALSE. Problem (1) gives a counterexample.

(g.) Linear operators on infinite-dimensional vector spaces never have eigenvalues.

FALSE. The identity operator on any space has 1 as an eigenvalue (and all nonzero vectors as eigenvectors). The textbook gives a more interesting example.

(h.) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.

TRUE.

(i.) Similar matrices always have the same eigenvalues.

TRUE. This is because similar matrices represent the same linear transformation relative to different bases.

(j.) Similar matrices always have the same eigenvectors.

FALSE. This is because the coordinates of an eigenvector for a linear transformation are different in different bases.

(k.) The sum of two eigenvectors of an operator T is always an eigenvector of T.

FALSE. Again, consider problem (1). However, the sum of two eigenvectors corresponding to the same eigenvalue is always either the zero vector or an eigenvector.

(l.) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

FALSE. Problem (1) gives a counterexample.

(m.) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.

FALSE. Problem (1) gives a counterexample.

(n.) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.

FALSE. The exception is the zero vector

(o.) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

TRUE.

(p.) Let $A \in M_{n\times n}(F)$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose j^{th} column is v_n $(1 \leq j \leq n)$, then $Q^{-1}AQ$ is a diagonal matrix.

TRUE. $Q^{-1}AQ$ is the matrix of L_A in the basis β, which is a diagonal matrix.

(q.) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

FALSE. It must also be the case that the characteristic polynomial of T splits.

(r.) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

TRUE. The vector space has a basis consisting of eigenvectors, each of which corresponds to some eigenvalue.

(s.) You can always tell from the characteristic polynomial of A whether A is diagonalizable.

FALSE. Problems (1) and (3) from last time give an example of two matrices with the same characteristic polynomial, only one of which is diagonalizable.

(t.) You can sometimes tell from the characteristic polynomial of A whether A is diagonalizable.

TRUE. If the characteristic polynomial does not split, A is not diagonalizable. If the characteristic polynomial splits and each root has multiplicity 1, then A is diagonalizable.

(u.) You can always tell from the characteristic polynomial of A whether A is invertible.

TRUE. A is invertible if and only if the only solution to $Av = 0$ is $v = 0$, which happens if and only if 0 is not an eigenvalue of A, which happens if and only if 0 is not a root of the characteristic polynomial of A, which happens if and only if the characteristic polynomial of A has a nonzero constant term.

(2.) Find an invertible matrix Q and find a diagonalizable matrix B such that either $QAQ^{-1} = B$ or $Q^{-1}AQ = B$. Be sure to say which of these two equations holds for your Q and B.

$$A = \begin{pmatrix} 1 & -7 & 2 \\ 0 & 2 & 0 \\ 0 & -10 & 2 \end{pmatrix}.$$

There was an error in this problem; this matrix actually is not diagonalizable. The characteristic polynomial is $det \begin{pmatrix} 1-t & -7 & 2 \\ 0 & 2-t & 0 \\ 0 & -10 & 2-t \end{pmatrix} = (1-t)(2-t)(2-t)$, which has two roots, 1 (multiplicity 1) and 2 (multiplicity 2).

To find an eigenvector corresponding to $\lambda = 1$, solve the equation $(A - I)v = 0$, or $\begin{pmatrix} 0 & -7 & 2 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ to get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$, so an eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

To find an eigenvector corresponding to $\lambda = 2$, solve the equation $(A - 2I)v = 0$, or $\begin{pmatrix} -1 & -7 & 2 \\ 0 & 0 & 0 \\ 0 & -10 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ to get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ 0 \\ x_3 \end{pmatrix}$, so an eigenvector is $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. There are not two linearly independent eigenvectors.

Since the eigenvalue 2 has multiplicity 2, and the corresponding eigenspace has dimension 1, A is not diagonalizable.

(3.) For the matrix A in problem (2), find a basis for the eigenspace of A corresponding to each eigenvalue. Describe each of these eigenspaces geometrically. (Be specific. Don't just say "a line"; specify which line.)

The eigenspace corresponding to $\lambda = 1$ is the line through the origin in the direction of the eigenvector $(1, 0, 0)$, or the x-axis. It is described by the pair of equations $y = 0$ and $z = 0$.

The eigenspace corresponding to $\lambda = 2$ is the line through the origin in the direction of the eigenvector $(2, 0, 1)$. It is described by the pair of equations $y = 0$ and $x = 2z$.

(4.) Test the matrix A for diagonalizability.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial $(1 - t)(1 - t)(2 - t)$ splits. The eigenvalues are 1, with multiplicity 2, and 2, with multiplicity 1. A is diagonalizable, then, if and only if the eigenspace corresponding to eigenvalue $\lambda = 1$ has dimension 2.

To find out, solve the equation $(A - I)v = 0$, or $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ to get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$. The eigenspace has dimension 1, so A is not diagonalizable.

(5.) Suppose a linear operator T on an n-dimensional vector space V has only one eigenvalue $\lambda = 1$, and T is diagonalizable. What can you conclude about T?

What can you say in general about diagonalizable linear operators with a single eigenvalue?

Since T is diagonalizable, there is a basis β consisting of eigenvectors of T. Since the only eigenvalue of T is 1, for each eigenvector $v \in \beta$, we must have $T(v) = v$. Since a linear transformation is determined by its values on a basis, T must be the identity transformation.

If the only eigenvalue of T is λ, the same argument applies, except now $T(v) = \lambda v$ for every v, and so $T = \lambda I$, where I is the identity transformation.

(6.) Show that if T is a diagonalizable linear operator on an n-dimensional vector space V with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then each vector v in V can be expressed uniquely as

$$v = v_1 + v_2 + \cdots + v_k$$

where $v_i \in E_{\lambda_i}$.

Since T is diagonalizable, there is a basis for V consisting of eigenvectors for T. Say $\beta = \{v_{1,1}, v_{1,2}, \dots, v_{1,m_1}, v_{2,1}, v_{2,2}, \dots, v_{2,m_2}, \dots, v_{k,1}, v_{k,2}, \dots, v_{k,m_k}\}$, where each $v_{i,j}$ is an eigenvector corresponding to eigenvalue λ_i, so $v_{i,j} \in E_{\lambda_i}$.

Since β is a basis, any $v \in V$ can be written as

$$v = \sum_{i=1}^{k} \sum_{j=1}^{m_i} a_{i,j} v_{i,j}.$$

Since $v_{i,j} \in E_{\lambda i}$, we have

$$\sum_{j=1}^{m_i} a_{i,j} v_{i,j} \in E_{\lambda_i}.$$

This shows we can express v as the sum of elements $v_1, v_2, \dots, v_k$ where $v_i = \sum_{j=1}^{m_i} a_{i,j} v_{i,j} \in E_{\lambda_i}$.

To show this expression is unique, suppose we have

$$v = v_1 + v_2 + \cdots + v_k = w_1 + w_2 + \cdots + w_k$$

where $v_i, w_i \in E_{\lambda_i}$. We must show $v_i = w_i$ for all i. We have

$$(v_1 + v_2 + \cdots + v_k) - (w_1 + w_2 + \cdots + w_k) = (v_1 - w_1) + (v_2 - w_2) + \cdots + (v_n - w_n) = 0,$$

where $(v_i - w_i) \in E_{\lambda_i}$. If any of the $(v_i - w_i)$ are nonzero, we have a contradiction to Theorem 5.5 (eigenvectors that correspond to distinct eigenvalues are linearly independent). Therefore $v_i - w_i = 0$ for all i, so $v_i = w_i$ for all i.