Math 24
Winter 2010
Friday, February 19

(1.) Find the eigenvalues, and a basis for each of the corresponding eigenspaces, of the matrix $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$. Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of this matrix?

The characteristic polynomial of this matrix is

$$det \begin{pmatrix} -t & 1 & -1 \\ 0 & 1-t & 0 \\ -1 & 1 & -t \end{pmatrix} = (1-t)(t^2-1) = (1-t)(t-1)(t+1).$$

(The easy way to compute this is cofactor expansion along the second row.) It has two roots, $\lambda = 1$ (a double root) and $\lambda = -1$ (a single root).

To find eigenvectors of A associated with the eigenvalue λ, we wish to solve the matrix equation $Ax = \lambda x$, which can be rewritten as $Ax = \lambda(Ix)$, or $(A - \lambda I)x = 0$. To do this, we can row reduce the matrix $A - \lambda I$.

For $\lambda = -1$, we row reduce $\begin{pmatrix} 0-(-1) & 1 & -1 \\ 0 & 1-(-1) & 0 \\ -1 & 1 & 0-(-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ to get $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. This is the coefficient matrix of the homogeneous system of linear equations

$$x_1 - x_3 = 0$$
$$x_2 = 0$$
$$0 = 0$$

The first equation begins with x_1, and can be used to determine the value of x_1:

$$x_1 = x_3$$

The second equation begins with x_2, and can be used to determine the value of x_2

$$x_2 = 0.$$

There is no equation remaining to determine the value of x_3, so x_3 can be anything. Because there is one variable whose value is not determined by an equation, we will have one *parameter* in the solution set, and the solution set will be one-dimensional.

Set $x_3 = s$ (where the parameter s can be any real number). Then the values of the other variables are determined as $x_1 = x_3 = s$, and $x_2 = 0$. The complete solution set to $(A - \lambda I)x = 0$, where A is the matrix of the problem and λ is -1, is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ s \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Since $(A-\lambda I)x=0$ is equivalent to $Ax=\lambda x$, the vector $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda=-1$, and $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ is a basis for the eigenspace of A corresponding to the eigenvalue $\lambda=-1$.

For $\lambda=1$, we row reduce $\begin{pmatrix}0-1 & 1 & -1\\ 0 & 1-1 & 0\\ -1 & 1 & 0-1\end{pmatrix}=\begin{pmatrix}-1 & 1 & -1\\ 0 & 0 & 0\\ -1 & 1 & -1\end{pmatrix}$ to get $\begin{pmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$.
This is the coefficient matrix of the homogeneous system of linear equations

$x_1-x_2+x_3=0$
$0=0$
$0=0$

The first equation begins with x_1, and can be used to determine the value of x_1:

$x_1=x_2-x_3$

There is no equation remaining to determine the value of x_2 or of x_3, so x_2 and x_3 can be anything. Because there are two variables whose value is not determined by an equation, we will have two *parameters* in the solution set, and the solution set will be two-dimensional.

Set $x_2=s$ and $x_3=t$ (where the parameters s and t can be any real numbers). Then the value of the other variable is determined as $x_1=x_2-x_3=s-t$. The complete solution set to $(A-\lambda I)x=0$, where A is the matrix of the problem and λ is 1, is

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}s-t\\s\\t\end{pmatrix}=s\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Since $(A-\lambda I)x=0$ is equivalent to $Ax=\lambda x$, the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ are eigenvectors of A corresponding to the eigenvalue $\lambda=1$, and $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ is a basis for the eigenspace of A corresponding to the eigenvalue $\lambda=1$.

Since we found three linearly independent eigenvectors, $\beta=\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ forms a basis for $\mathbb{R}^3$ consisting of eigenvectors of A.

A diagonal matrix similar to A is $[L_A]_\beta$. Note that if α is the standard ordered basis, then $A=[L_A]_\alpha=Q[L_A]_\beta Q^{-1}$ where Q is the change of coordinate matrix that changes β coordinates to α coordinates.