Math 24
Winter 2010
Wednesday, February 17

(1.) TRUE or FALSE?

(a.) If E is an elementary matrix, then $det(E) = \pm 1$.

(b.) For any $A, B \in M_{n\times n}(F)$, $det(AB) = (det(A))(det(B))$.

(c.) A matrix $A \in M_{n\times n}(F)$ is invertible if and only if $det(A) = 0$.

(d.) A matrix $A \in M_{n\times n}(F)$ has rank n if and only if $det(A) \neq 0$.

(e.) For any $A \in M_{n\times n}(F)$, $det(A^t) = -det(A)$.

(f.) The determinant of a square matrix can be evaluated by cofactor expansion along any column.

(g.) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

(h.) Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{det(M_k)}{det(A)} \text{ for } k = 1, 2, \ldots, n.$$

(i.) If Q is an invertible matrix, then $det(Q^{-1}) = \dfrac{1}{det(Q)}$.

(j.) The determinant of a lower triangular $n \times n$ matrix is the product of its diagonal entries. (A matrix is lower triangular if the only nonzero entries are on or below the main diagonal.)

(2.) Let A be an $n \times n$ matrix, and k a scalar. Find the determinant of kA in terms of the determinant of A.

(3.) Show that if A and B are similar $n \times n$ matrices, then $det(A) = det(B)$.

(4.) Suppose that $M \in M_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},$$

where A is a square matrix, 0 is a zero matrix, and I is an $m \times m$ identity matrix. Prove that $det(M) = det(A)$.

(5.) Let $A \in M_{n\times n}(F)$ be nonzero. For any m with $1 \leq m \leq n$, an $m \times m$ submatrix is obtained by deleting $n-m$ rows and $n-m$ columns of A. For example, if we start with $A = \begin{pmatrix} 1 & 1 & 1 & 4 \\ 2 & 3 & 1 & 8 \\ -2 & 0 & 0 & -4 \\ 1 & 4 & 4 & 10 \end{pmatrix}$ and delete rows 2 and 3 and columns 2 and 4, we get the 2×2 submatrix $\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$.

(a.) Show that if A is an $n \times n$ matrix and there is a $k \times k$ submatrix of A with nonzero determinant, then $rank(A) \geq k$.

(b.) Show that if A is an $n \times n$ matrix with rank k, then there is a $k \times k$ submatrix of A with nonzero determinant.