Math 24
Winter 2010
Monday, February 15

(1.) TRUE or FALSE?

(a.) The function $det : M_{n\times n}(F) \to F$ is a linear transformation.

FALSE. For example, $det(I_2) = 1$ but $det(2 \cdot I_2) = 4$.

(b.) The determinant of a $n \times n$ matrix is a linear function of each row of the matrix when the other rows are held fixed.

TRUE.

(c.) If $A \in M_{n\times n}(F)$ and $det(A) = 0$ then A is invertible.

FALSE. If A is not invertible, then $det(A) = 0$.

(d.) If u and v are vectors in $\mathbb{R}^2$ emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is $det \begin{pmatrix} u \\ v \end{pmatrix}$.

FALSE. This determinant may be negative. The area is the absolute value of the determinant.

(e.) A coordinate system is right-handed if and only if its orientation equals 1.

TRUE.

(f.) The determinant of a square matrix can be evaluated by cofactor expansion along any row.

TRUE.

(g.) If two rows of a square matrix A are identical, then $det(A) = 0$.

TRUE.

(h.) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $det(B) = det(A)$.

FALSE. The determinant of B is that scalar times the determinant of A.

(i.) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $det(B) = -det(A)$.

TRUE.

(j.) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $det(B) = k\, det(A)$.

FALSE. In this case $det(B) = det(A)$.

(k.) If $A \in M_{n\times n}(F)$ has rank n then $det(A) = 0$.

FALSE. If A has rank less than n then $det(A) = 0$.

(l.) The determinant of an upper triangular matrix equals the product of its diagonal entries.

TRUE.

(2.) Evaluate the determinant of the following matrix, first by cofactor expansion along any row, second by using elementary row operations to transform it to an upper triangular matrix.

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$$

If we expand along the first row, we get

$$det \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix} = (0)det \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} - (1)det \begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix} + (2)det \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} =$$

$$(0)(9) - (1)(6) + (2)(-3) = -12.$$

Now we perform the following sequence of elementary row operations: Interchange rows 1 and 2. Add (2) times row 1 to row 3. Add (-3) times row 2 to row 3. This gives the following transformation:

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 3 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{pmatrix}.$$

The determinant of this upper triangular matrix is the product of its diagonal entries, namely 12. We performed one interchange of rows, which changes the sign of the determinant, so the determinant of our original matrix was -12.

Since we got the same answer each time, our arithmetic is probably right.

For the remaining problems, let $G : M_{n\times n}(F) \to F$ be any function such that

(a.) G is a linear function of any row, when the other rows are held fixed.
(b.) If two rows of a matrix A are identical, then $G(A) = 0$.
(c.) $G(I_n) = 1$.

Show the following:

(3.) If B is obtained from A by multiplying row i by the scalar r, then $G(B) = r\,G(A)$. (Hint: Use the fact that G is a linear function of row i when the other rows are held fixed.)

First let us formally state what this assumption on G tells us. If $r_1, r_2, \ldots, r_n$ are the rows of A, r'_i is another row vector in F^n, and c is a scalar, then the two defining characteristics of a linear function tell us, if we write the matrix A in terms of its rows as $A = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$, that

$$G \begin{pmatrix} r_1 \\ \vdots \\ r_{i-1} \\ r_i + r'_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix} = G \begin{pmatrix} r_1 \\ \vdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix} + G \begin{pmatrix} r_1 \\ \vdots \\ r_{i-1} \\ r'_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix}, \text{ and}$$

$$G \begin{pmatrix} r_1 \\ \vdots \\ r_{i-1} \\ c\, r_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix} = c\, G \begin{pmatrix} r_1 \\ \vdots \\ r_{i-1} \\ r_i \\ r_{i+1} \\ \vdots \\ r_n \end{pmatrix}.$$

But this second condition is exactly what we are to prove.

(4.) If row i of A consists entirely of zeroes, then $G(A) = 0$.

If B is obtained from A by multiplying row i by 0, then by problem (3) we have $det(B) = 0\, det(A) = 0$. But if row i of A consists entirely of zeroes, then $B = A$, and so $det(A) = 0$.

(5.) If B is obtained from A by adding a scalar multiple of row i to row j, then $G(B) = G(A)$. (Hint: Use the fact that G is a linear function of row j when the other rows are held fixed.)

By linearity,

$$G(B) = G \begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ r_j + k\, r_i \\ r_{j+1} \\ \vdots \\ r_n \end{pmatrix} = G \begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ r_j \\ r_{j+1} \\ \vdots \\ r_n \end{pmatrix} + k\, G \begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ r_i \\ r_{j+1} \\ \vdots \\ r_n \end{pmatrix}.$$

In the last term, the i^{th} row and j^{th} row are identical (both are equal to r_i) and so, by condition (b) on G, we have

$$G(B) = G\begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ r_j + k\,r_i \\ r_{j+1} \\ \vdots \\ r_n \end{pmatrix} = G\begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ r_j \\ r_{j+1} \\ \vdots \\ r_n \end{pmatrix} + k\,0 = G\begin{pmatrix} r_1 \\ \vdots \\ r_{j-1} \\ r_j \\ r_{j+1} \\ \vdots \\ r_n \end{pmatrix} = G(A).$$

(6.) If B is obtained from A by interchanging row i and row j, then $G(B) = -G(A)$. (Hint: This type 1 elementary row operation can be accomplished by a combination of type 2 and type 3 operations.)

If the i^{th} of A is r_i and the j^{th} row is r_j, then B can be obtained from A by the following sequence of elementary row operations:

1. Add row i to row j: New j^{th} row is $r_i + r_j$, value of G is unchanged.
2. Add (-1) times row j to row i: New i^{th} row is $r_i - (r_i + r_j) = -r_j$, value of G is unchanged.
3. Add row i to row j: New j^{th} row is $r_i + r_j + (-r_j) = r_i$, value of G is unchanged. The j^{th} row is now r_i.
4. Multiply row i by (-1): New i^{th} row is $(-1)(-r_j) = r_j$, value of G is multiplied by -1. The i^{th} row is now r_j, and the transformation is completed.

The net result is to interchange rows i and j and to change the sign of the determinant.

(7.) $G(A) = det(A)$ for any $n \times n$ matrix A. (Hint: A can be transformed by elementary row operations to either I_n or a matrix with a row of zeroes.)