

Math 24
Winter 2010

Here is an example of a proof, written up in complete sentences, with every step explained.

Proposition: If F is any field, and x is any element of F , then $x \cdot 0 = 0$.

Proof: Let x be any element of the field F . We will show $x \cdot 0 = 0$.

Because 0 is an additive identity element, we have

$$0 + 0 = 0.$$

Multiplying both sides by x , we get

$$x \cdot (0 + 0) = x \cdot 0.$$

Applying the distributive property, we get

$$x \cdot 0 + x \cdot 0 = x \cdot 0.$$

Adding the additive inverse of $x \cdot 0$ to both sides, we get

$$(x \cdot 0 + x \cdot 0) + (-(x \cdot 0)) = x \cdot 0 + (-(x \cdot 0)),$$

and by the associative property of addition,

$$x \cdot 0 + (x \cdot 0 + (-(x \cdot 0))) = x \cdot 0 + (-(x \cdot 0)).$$

By the definition of additive inverse, $x \cdot 0 + (-(x \cdot 0)) = 0$, so we get

$$x \cdot 0 + 0 = 0.$$

Finally, because 0 is an additive identity element, $x \cdot 0 + 0 = x \cdot 0$, and so we get

$$x \cdot 0 = 0.$$

Note: In this proof I was explicit about every use of the axioms for a field, such as the associative property of addition. This is because abstract fields are new to us, and it's important to point out that we are using only the axioms, and not things we know about \mathbb{R} that may or may not be true about weirder fields. (For example, that $1 + 1 \neq 0$.) In practice, we are familiar with the basic properties of fields, and we will feel free to use them without comment.

The same is true of the vector space axioms,. When we are first proving basic facts about vector spaces, we will note every use of the vector space axioms. After we are more familiar with them (which will be anytime after Section 1.2), we will use basic properties, like associativity and additive inverses, without comment.

A string of equations with no explanation, such as

$$\begin{aligned}
 0 + 0 &= 0 \\
 x \cdot (0 + 0) &= x \cdot 0 \\
 x \cdot 0 + x \cdot 0 &= x \cdot 0 \\
 (x \cdot 0 + x \cdot 0) + (-(x \cdot 0)) &= x \cdot 0 + (-(x \cdot 0)) \\
 x \cdot 0 + (x \cdot 0 + (-(x \cdot 0))) &= x \cdot 0 + (-(x \cdot 0)) \\
 x \cdot 0 + 0 &= 0 \\
 x \cdot 0 &= 0,
 \end{aligned}$$

is not a proof. Proofs are written in English, which means you must use complete sentences, and always explain enough so your reader can follow your reasoning without working too hard.

A string of equations with short explanations of each step can, however, be part of a proof. A two-column proof of the sort you probably learned to do in high school can also be part of a proof. The important question is what is most readable and easiest to follow.

Here is an alternative version of the same proof. Note that this style is not always appropriate, and in any case, you can't do without the preliminaries like stating the proposition you are proving, and saying what x represents. The more complex your proof, the more explanation your reasoning needs.

Proposition: If F is any field, and x is any element of F , then $x \cdot 0 = 0$.

Proof: Let x be any element of the field F . The following two-column proof, with axiom numbers taken from Appendix C of the textbook, shows that $x \cdot 0 = 0$.

$0 + 0 = 0$	(F3, additive identity)
$x \cdot (0 + 0) = x \cdot 0$	(multiply both sides by x)
$x \cdot 0 + x \cdot 0 = x \cdot 0$	(F5, distributivity)
$(x \cdot 0 + x \cdot 0) + (-(x \cdot 0)) = x \cdot 0 + (-(x \cdot 0))$	(F4, additive inverse)
$x \cdot 0 + (x \cdot 0 + (-(x \cdot 0))) = x \cdot 0 + (-(x \cdot 0))$	(F2, associativity)
$x \cdot 0 + 0 = 0$	(F4, additive inverse)
$x \cdot 0 = 0$	(F3, additive identity).