Math 24
Winter 2010
Monday, February 8

(1.) TRUE or FALSE?

(a.) The rank of a matrix is equal to the number of its nonzero columns.

FALSE. It is the maximum number of linearly independent columns.

(b.) The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

FALSE. The rank cannot be larger than this, but it can be smaller.

(c.) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.

TRUE.

(d.) Elementary row operations preserve rank.

TRUE. Elementary row operations do not change $N(L_A)$, so they preserve the nullity, and therefore the rank, of L_A.

(e.) Elementary column operations do not necessarily preserve rank.

FALSE. Elementary column operations do not change $R(L_A)$, so they preserve the rank of L_A.

(f.) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

TRUE. A and A^t have the same rank.

(g.) The inverse of a matrix can be computed exclusively by means of elementary row operations.

TRUE. It can also, alternatively, be computed by means of elementary column operations.

(h.) The rank of an $m \times n$ matrix is at most the smaller of m and n.

TRUE. It can, however, be smaller.

(i.) An $n \times n$ matrix having rank n is invertible.

TRUE. If the rank of L_A is n, then L_A is invertible, and so A is invertible.

(2.) For each matrix, find the rank, and compute the inverse (if it exists):

We try to transform A to I using elementary row operations, and simultaneously perform the same elementary row operations on I to transform I to A^{-1}. Some of the steps shown here involve two elementary row operations.

(a.) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 1 & 3 & 4 & | & 0 & 1 & 0 \\ 2 & 3 & -1 & | & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & -1 & 1 & 0 \\ 0 & -1 & -3 & | & -2 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & -5 & | & 3 & -2 & 0 \\ 0 & 1 & 3 & | & -1 & 1 & 0 \\ 0 & 0 & 0 & | & -3 & 1 & 1 \end{pmatrix}.$$

The transformed matrix on the left has two linearly independent columns, so the rank of the original matrix is 2, and it is not invertible.

(b.) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$

$$\begin{pmatrix} 0 & -2 & 4 & | & 1 & 0 & 0 \\ 1 & 1 & -1 & | & 0 & 1 & 0 \\ 2 & 4 & -5 & | & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & -2 & 4 & | & 1 & 0 & 0 \\ 2 & 4 & -5 & | & 0 & 0 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & -2 & 4 & | & 1 & 0 & 0 \\ 0 & 2 & -3 & | & 0 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & -2 & | & -\frac{1}{2} & 0 & 0 \\ 0 & 2 & -3 & | & 0 & -2 & 1 \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} 1 & 0 & 1 & | & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -2 & | & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & | & -\frac{1}{2} & 3 & -1 \\ 0 & 1 & 0 & | & \frac{3}{2} & -4 & 2 \\ 0 & 0 & 1 & | & 1 & -2 & 1 \end{pmatrix}.$$

The original matrix was transformed to the identity matrix, so its rank is 3, and the transformed matrix on the right is its inverse.

(3.) Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B such that $AB = I_m$. (Hint: Think about the linear transformation L_A.)

If B is an $n \times m$ matrix, then $L_{AB} = L_A L_B$ is a function from F^m to F^m. If L_{AB} is the identity transformation, then AB is the identity matrix. Therefore, we want to find a matrix B such that $L_A L_B$ is the identity transformation.

The linear transformation $L_A : F^n \to F^m$ has rank m (because the rank of A is the rank of L_A). Because the rank of L_A equals the dimension of the codomain, L_A is onto. This means every vector in F^m is in the range.

In particular, if $e_1, e_2, \ldots, e_m$ are the standard basis vectors of F^m, we can find $v_1, v_2, \ldots, v_m$ in F^n such that, for all i, we have $e_i = L_A(v_i)$.

Now let $T : F^m \to F^n$ be the linear transformation such that $T(e_i) = v_i$. (We know there is such a linear transformation because the e_i form a basis.) There is an $n \times m$ matrix B such that $T = L_B$. (The matrix B is the matrix representing T relative to the standard bases.) Hence, for all i, we have $L_B(e_i) = T(e_i) = v_i$.

Now, for all i we have $L_A(L_B(e_i)) = L_A(v_i) = e_i$. That is, the composition $L_A L_B$ sends each basis vector to itself. Because a linear transformation is determined by what it does to the basis vectors, this means $L_A L_B$ is the identity transformation, and we are done.