Math 24
Winter 2010
Monday, February 1

(1.) TRUE or FALSE?

(a.) Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β' coordinates into β coordinates. Then the j^{th} column of Q is $[x_j]_{\beta'}$.

FALSE. The j^{th} column of Q is $[x'_j]_{\beta}$.

(b.) Every change of coordinate matrix is invertible.

TRUE. The inverse is the matrix that changes coordinates back again.

(c.) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β' coordinates into β coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

TRUE.

(d.) The matrices $A, B \in M_{n\times n}(F)$ are called similar if $B = Q^t AQ$ for some $Q \in M_{n\times n}(F)$.

FALSE. A and B are similar if $B = Q^{-1}AQ$ for some $Q \in M_{n\times n}(F)$.

(e.) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$.

TRUE. This is why similar matrices are interesting.

(f.) Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β' coordinates into β coordinates. Then $Q = [I_V]_{\beta'}^{\beta}$.

TRUE. This is how we arrive at change of coordinate matrices.

(g.) Every invertible matrix is a change of coordinate matrix.

TRUE. If A is invertible, so is L_A, and therefore the columns of A, which span the range of L_A, must be linearly independent. Then A is the matrix that changes β coordinates into standard coordinates in F^n, where β is the ordered basis consisting of the columns of A.

For the next problems you may use the following fact (which you can check by multiplying these matrices together): If $ad - bc \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Please note, I don't guarantee that the arithmetic in these solutions is always correct.

(2.) Let α be the standard ordered basis for $\mathbb{R}^2$, β be the ordered basis $\{(1,2),\ (2,-1)\}$, and γ be the ordered basis $\{(1,-1),\ (1,1)\}$. Write down the change of coordinate matrices for changing:

β coordinates into α coordinates.

The columns of this matrix are the α coordinates of the elements of the basis β. Since α is the standard basis, we can just write this one down.

$$Q_\beta^\alpha = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

α coordinates into β coordinates.

$$Q_\alpha^\beta = (Q_\beta^\alpha)^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}.$$

γ coordinates into α coordinates.

$$Q_\gamma^\alpha = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

α coordinates into γ coordinates.

$$Q_\alpha^\gamma = (Q_\gamma^\alpha)^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

β coordinates into γ coordinates.

To change from β coordinates to γ coordinates, we can first change from β coordinates to α coordinates, and then from α coordinates to γ coordinates.

$$Q_\beta^\gamma = Q_\alpha^\gamma Q_\beta^\alpha = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

γ coordinates into β coordinates.

$$Q_\gamma^\beta = \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{1}{5} \end{pmatrix}.$$

(3.) Here α, β, and γ are the same ordered bases for $\mathbb{R}^2$ as in problem (2).

(a.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(a,b) = (a+b, a-b)$. Write down the matrices $[T]_\alpha$, $[T]_\beta$, $[T]_\gamma$, and $[T]_\beta^\gamma$.

$T(1,0) = (1,1)$ and $T(0,1) = (1,-1)$, so the matrix of T in the standard basis is

$$[T]_\alpha = [T]_\alpha^\alpha = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$[T]_\beta = [T]_\beta^\beta = Q_\alpha^\beta [T]_\alpha^\alpha Q_\beta^\alpha = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{7}{5} \\ \frac{7}{5} & -\frac{1}{5} \end{pmatrix}$$

$$[T]_\gamma = [T]_\gamma^\gamma = Q_\alpha^\gamma [T]_\alpha^\alpha Q_\gamma^\alpha = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$[T]_\beta^\gamma = [T]_\gamma^\gamma Q_\beta^\gamma = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

(b.) Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. Write down the matrices $[L_A]_\alpha$, $[L_A]_\beta$, $[L_A]_\gamma$, and $[L_A]_\beta^\gamma$.

Since α is the standard basis, $[L_A]_\alpha = A$. We can find $[L_A]_\beta$, $[L_A]_\gamma$, and $[L_A]_\beta^\gamma$ in the same way as in part (a).

(4.) If $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$ are ordered bases for $P_2(\mathbb{R})$, find the change of coordinate matrix that changes β' coordinates into β coordinates.

The column of this matrix are $[1]_\beta$, $[x]_\beta$, and $[x^2]_\beta$. To find the first column, we solve $1 = a(2x^2 - x) + b(3x^2 + 1) + cx^2$ to get $a = 0$, $b = 1$, $c = -3$, and so $[1]_\beta = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$. Finding $[x]_\beta$ and $[x^2]_\beta$ in the same way, we get $Q_{\beta'}^\beta = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{pmatrix}$.