Math 24
Winter 2010
Friday, January 29

(1.) TRUE or FALSE? In each part, V and W are finite-dimensional vector spaces with ordered bases α and β respectively, $T : V \to W$ is linear, and A and B denote matrices.

(a.) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$.

FALSE. Even if T were invertible, we would have $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\beta^\alpha$.

(b.) T is invertible if and only if T is one-to-one and onto.

TRUE. This is true for all functions, not just linear transformations.

(c.) $T = L_A$ where $A = [T]_\alpha^\beta$.

FALSE. T can be *represented* by L_A where $A = [T]_\alpha^\beta$, but T can also be represented by L_B, where $B = [T]_\gamma^\delta$ for some other ordered bases γ and δ.

(d.) $M_{2\times 3}(F)$ is isomorphic to F^5.

FALSE. $M_{2\times 3}(F)$ is isomorphic to F^6.

(e.) $P_n(F)$ is isomorphic to $P_m(F)$ if and only if $n = m$.

TRUE. Finite dimensional vector spaces over F are isomorphic if and only if they have the same dimension.

(f.) $AB = I$ implies that A and B are invertible.

FALSE. But if A and B are square, then $AB = I$ implies that A and B are invertible.

(g.) If A is invertible, then $(A^{-1})^{-1} = A$.

TRUE.

(h.) A is invertible if and only if L_A is invertible.

TRUE. In this case $(L_A)^{-1} = L_{A^{-1}}$.

(i.) A must be square in order to possess an inverse.

TRUE, just as L_A must have $dim(domain(L_A)) = dim(codomain(L_A))$ in order to (possibly) be invertible.

(2.) For each linear transformation T, determine whether T is invertible and justify your answer.

(a.) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (a - 2b, b, 3a + 4b)$.

NO, $dim(domain(T)) \neq dim(codomain(T))$.

(b.) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (3a - 2, b, 4a)$.

NO, $dim(domain(T)) \neq dim(codomain(T))$.

(c.) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a,b,c) = (3a-2c, b, 3a+4b)$.

YES, $dim(domain(T)) = dim(codomain(T))$ and we can check that $N(T) = \{0\}$.

(d.) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$.

NO, $dim(domain(T)) \neq dim(codomain(T))$.

(e.) $T : M_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.

NO, although $dim(domain(T)) = dim(codomain(T))$, T is not onto because $x^3 \notin R(T)$.

(f.) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

YES, $dim(domain(T)) = dim(codomain(T))$ and we can check that $N(T) = \{0\}$.

(3.) Let $V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \middle| a,b,c \in F \right\}$. Construct an isomorphism from V to F^3.

Solution 1: $T\begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a,b,c)$. This is linear and clearly onto, so as $dim(V) = dim(F^3)$, it must be an isomorphism.

Solution 2: Notice that V is actually the space of all 2×2 upper triangular matrices, so we can define T by $T\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (a,b,c)$.

To get another solution: Choose your favorite three linearly independent vectors in F^3, v_1, v_2, and v_3, and your favorite three linearly independent upper triangular 2×2 matrices, w_1, w_2 and w_3, and define $T(aw_1 + bw_2 + cw_3) = av_1 + bv_2 + cv_3$.

(4.) Let α be the standard ordered basis for $\mathbb{R}^2$, and β be the ordered basis $\{(1,2),(2,-1)\}$.

(a.) Compute $[(1,0)]_\beta$, $[(0,1)]_\beta$, $[(1,2)]_\alpha$, and $[(2,-1)]_\alpha$.

To find $[(1,0)]_\beta = \begin{pmatrix} a \\ b \end{pmatrix}$, solve $(1,0) = a(1,2) + b(2,-1)$ to get $a = \frac{1}{5}$ and $a = \frac{2}{5}$, so $[(1,0)]_\beta = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$. The same method gives $[(0,1)]_\beta = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}$.

$[(1,2)]_\alpha$ expresses $(1,2)$ in the standard basis, so $[(1,2)]_\alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $[(2,-1)]_\alpha = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b.) Let $\mathbb{I} = I_{\mathbb{R}^2}$ be the identity transformation on $\mathbb{R}^2$ defined by $\mathbb{I}(v) = v$. Compute $[\mathbb{I}]_\alpha^\beta$ and $[\mathbb{I}]_\beta^\alpha$.

If $(1,0)$ and $(0,1)$ are the vectors of the standard ordered basis α, then the columns of $[\mathbb{I}]_\alpha^\beta$ are the coordinates of $\mathbb{I}(1,0) = (1,0)$ and $\mathbb{I}(0,1) = (0,1)$ in the ordered basis β, or $[(1,0)]_\beta$ and $[(0,1)]_\beta$, which we found in part (a): $[\mathbb{I}]_\alpha^\beta = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix}$.

In the same way, $[\mathbb{I}]_\beta^\alpha = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

(c.) Because composition of linear transformations corresponds to multiplication of their matrices, we should have that $[\mathbb{I}]_\alpha^\beta[\mathbb{I}]_\beta^\alpha = [\mathbb{I}]_\beta^\beta = I$, where I is the 2×2 identity matrix. Check this by multiplying your matrices together.

$$\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice what these matrices are. The column vector $[v]_\alpha$ represents the coordinates of v in basis α, and the column vector $[\mathbb{I}]_\alpha^\beta[v]_\alpha = [\mathbb{I}(v)]_\beta = [v]_\beta$ represents the coordinates of v in basis β. If $v = (x, y)$, then $\begin{pmatrix} x \\ y \end{pmatrix}$ gives the coordinates of v in the standard basis α, and the matrix product $[\mathbb{I}]_\alpha^\beta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$ gives the coordinates of v in the new basis β (which means that $v = s(1, 2) + t(2, -1)$). In other words, multiplying by the matrix $[\mathbb{I}]_\alpha^\beta$ changes from standard coordinates to β-coordinates, and multiplying by the inverse matrix $[\mathbb{I}]_\beta^\alpha$ changes from β-coordinates back to standard coordinates.

(d.) Notice that the vectors of β are perpendicular to each other. If the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is perpendicular projection onto the line through the origin in the direction of the vector $(1, 2)$, we can express T by the formula $T(s(1, 2) + t(2, -1)) = s(1, 2)$. Find the matrix $[T]_\alpha$ (that is, $[T]_\alpha^\alpha$) that represents T in the standard basis.

Hint: We can think of T as the three-step composition $\mathbb{I}\,T\,\mathbb{I}$ where $\mathbb{I}$ is the identity transformation on $\mathbb{R}^2$. You already found $[\mathbb{I}]_\alpha^\beta$ and $[\mathbb{I}]_\beta^\alpha$, and it's easy to write down $[T]_\beta$.

$$[T(1,2)]_\beta = [(1,2)]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } [T(2,-1)]_\beta = [(0,0)]_\beta = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ so } [T]_\beta^\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$$[T]_\alpha^\alpha = [\mathbb{I}]_\beta^\alpha [T]_\beta^\beta [\mathbb{I}]_\alpha^\beta = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}.$$

(5.) Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, let $\beta = \{w_1, w_2, \ldots, w_m\}$ be a basis for V, and for any $i \leq n$ and $j \leq m$ define a linear transformation $T_{ij} : V \to W$ by

$$T_{ij}(v_k) = \begin{cases} w_j & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

Show that $\{T_{ij} \mid i \leq n \text{ and } j \leq m\}$ is a basis for $\mathcal{L}(V, W)$. (Hint: Show the isomorphism $\mathcal{R} : \mathcal{L}(V, W) \to M_{m \times n}(F)$ defined by $\mathcal{R}(T) = [T]_\alpha^\beta$ takes this set to a basis for $M_{m \times n}(F)$.)

$\mathcal{R}(T_{ij})$ is the matrix A with $A_{ij} = 1$ and all other entries 0, so this set maps to the standard basis for $M_{m \times n}(F)$.