Math 24

Winter 2010

Friday, January 29

(1.) TRUE or FALSE? In each part, V and W are finite-dimensional vector spaces with ordered bases α and β respectively, $T : V \to W$ is linear, and A and B denote matrices.

(a.) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$.

(b.) T is invertible if and only if T is one-to-one and onto.

(c.) $T = L_A$ where $A = [T]_\alpha^\beta$.

(d.) $M_{2\times 3}(F)$ is isomorphic to F^5.

(e.) $P_n(F)$ is isomorphic to $P_m(F)$ if and only if $n = m$.

(f.) $AB = I$ implies that A and B are invertible.

(g.) If A is invertible, then $(A^{-1})^{-1} = A$.

(h.) A is invertible if and only if L_A is invertible.

(i.) A must be square in order to possess an inverse.

(2.) For each linear transformation T, determine whether T is invertible and justify your answer.

(a.) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (a - 2b, b, 3a + 4b)$.

(b.) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a, b) = (3a - 2, b, 4a)$.

(c.) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a, b, c) = (3a - 2c, b, 3a + 4b)$.

(d.) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$.

(e.) $T : M_{2\times 2}(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$.

(f.) $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + b & a \\ c & c + d \end{pmatrix}$.

(3.) Let $V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} \middle| a, b, c \in F \right\}$. Construct an isomorphism from V to F^3.

There are three very important isomorphisms we have seen so far. We also know a few important facts about creating isomorphisms.

(a.) If V is any vector space, the identity transformation $I_V : V \to V$ is an isomorphism.

(b) If V is any n-dimensional vector space over F, and β is a basis for V, the "coordinate coding" function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(v) = [v]_\beta$ is an isomorphism.

(c.) If V and W are n- and m- dimensional vector spaces over F with ordered bases α and β respectively, the "representation by matrix" function $\mathcal{R} : \mathcal{L}(V, W) \to M_{m\times n}(F)$ defined by $\mathcal{R}(T) = [T]_\alpha^\beta$ is an isomorphism.

(d.) If V and W are n-dimensional vector spaces over F, $\alpha = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, and $\{w_1, w_2, \ldots, w_n\}$ is a basis for W, then there is a unique isomorphism $T : V \to W$ such that $T(v_1) = w_1, T(v_2) = w_2, \ldots T(v_n) = w_n$.

(e.) If $T : V \to W$ and $U : W \to Z$ are isomorphisms, the composite (or composition) $UT : V \to Z$ is also an isomorphism.

(f.) If $T : V \to W$ is an isomorphism, then T is invertible, and its inverse $T^{-1} : W \to V$ is also an isomorphism.

Make sure you understand these things. They will be useful.

(4.) Let α be the standard ordered basis for $\mathbb{R}^2$, and β be the ordered basis $\{(1, 2), (2, -1)\}$.

(a.) Compute $[(1, 0)]_\beta$, $[(0, 1)]_\beta$, $[(1, 2)]_\alpha$, and $[(2, -1)]_\alpha$.

(b.) Let $\mathbb{I} = I_{\mathbb{R}^2}$ be the identity transformation on $\mathbb{R}^2$ defined by $\mathbb{I}(v) = v$. Compute $[\mathbb{I}]_\alpha^\beta$ and $[\mathbb{I}]_\beta^\alpha$.

(c.) Because composition of linear transformations corresponds to multiplication of their matrices, we should have that $[\mathbb{I}]_\alpha^\beta[\mathbb{I}]_\beta^\alpha = [\mathbb{I}]_\beta^\beta = I$, where I is the 2×2 identity matrix. Check this by multiplying your matrices together.

Notice what these matrices are. The column vector $[v]_\alpha$ represents the coordinates of v in basis α, and the column vector $[\mathbb{I}]^\beta_\alpha [v]_\alpha = [\mathbb{I}(v)]_\beta = [v]_\beta$ represents the coordinates of v in basis β. If $v = (x, y)$, then $\begin{pmatrix} x \\ y \end{pmatrix}$ gives the coordinates of v in the standard basis α, and the matrix product $[\mathbb{I}]^\beta_\alpha \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$ gives the coordinates of v in the new basis β (which means that $v = s(1,2) + t(2,-1)$). In other words, multiplying by the matrix $[\mathbb{I}]^\beta_\alpha$ changes from standard coordinates to β-coordinates, and multiplying by the matrix $[\mathbb{I}]^\alpha_\beta$ changes from β-coordinates back to standard coordinates.

(d.) Notice that the vectors of β are perpendicular to each other. If the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is perpendicular projection onto the line through the origin in the direction of the vector $(1, 2)$, we can express T by the formula $T(s(1,2) + t(2,-1)) = s(1,2)$. Find the matrix $[T]_\alpha$ (that is, $[T]^\alpha_\alpha$) that represents T in the standard basis.

Hint: We can think of T as the three-step composition $\mathbb{I}\, T\, \mathbb{I}$ where $\mathbb{I}$ is the identity transformation on $\mathbb{R}^2$. You already found $[\mathbb{I}]^\beta_\alpha$ and $[\mathbb{I}]^\alpha_\beta$, and it's easy to write down $[T]_\beta$.

(5.) Let $\alpha = \{v_1, v_2, \ldots, v_n\}$ be a basis for V, let $\beta = \{w_1, w_2, \ldots, w_m\}$ be a basis for V, and for any $i \leq n$ and $j \leq m$ define a linear transformation $T_{ij} : V \to W$ by

$$T_{ij}(v_k) = \begin{cases} w_j & \text{if } k = i; \\ 0 & \text{if } k \neq i. \end{cases}$$

Show that $\{T_{ij} \mid i \leq n \text{ and } j \leq m\}$ is a basis for $\mathcal{L}(V, W)$. (Hint: Show the isomorphism $\mathcal{R} : \mathcal{L}(V, W) \to M_{m \times n}(F)$ defined by $\mathcal{R}(T) = [T]_\alpha^\beta$ takes this set to a basis for $M_{m \times n}(F)$.)