Math 24
Winter 2010
Wednesday, January 27

(1.) TRUE or FALSE? In each part, V, W, and Z denote finite-dimensional vector spaces with ordered bases α, β and γ respectively, $T : V \to W$ and $U : W \to Z$ denote linear transformations, and A and B denote matrices.

(a.) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$.

(b.) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$.

(c.) $[U(w)]_\beta = [U]_\alpha^\beta [w]_\beta$ for all $w \in W$.

(d.) $[I_V]_\alpha = I$.

(e.) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$.

(f.) $A^2 = I$ implies that $A = I$ or $A = -I$.

(g.) $T = L_A$ for some matrix A.

(h.) $A^2 = 0$ implies that $A = 0$, where 0 denotes the zero matrix.

(i.) $L_{A+B} = L_A + L_B$.

(j.) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

(2.) If $A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}$, which of the matrix products AB and BA is defined?

Find the second column of that matrix product.

(3.) Write down a matrix A such that $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - 2y + z \\ x - 2z \end{pmatrix}$.

(4.) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates counterclockwise around the origin by ninety degrees (so if v is on the positive x axis, then $T(v)$ is on the positive y-axis), and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects every point perpendicularly onto the x-axis. Let β be the standard basis for $\mathbb{R}^2$.

(a.) Find explicit expressions for $T(x, y)$ and $U(x, y)$. Use these to write explicit expressions for $UT(x, y)$ and $TU(x, y)$. (Recall that $UT(x, y)$ denotes $U(T(x, y))$.) Find $UT(1, 0)$, $UT(0, 1)$, $TU(1, 0)$ and $TU(0, 1)$. Use these values to write down the matrices $[UT]_\beta$ and $[TU]_\beta$.

(b.) Find $U(1, 0)$, $U(0, 1)$, $T(1, 0)$ and $T(0, 1)$. Use these values to write down the matrices $[U]_\beta$ and $[T]_\beta$.

(c.) Use matrix multiplication to compute $[U]_\beta[T]_\beta$ and $[T]_\beta[U]_\beta$. Compare with your answers to part (a); did you get what you should?

(d.) Compute the matrix product $[U]_\beta[T]_\beta \begin{pmatrix} x \\ y \end{pmatrix}$. This should give you $[UT(x,y)]_\beta$. Compare this with your answer to part (a); did you get what you should?

(5.) Find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ other than $I_{\mathbb{R}^2}$ and $-I_{\mathbb{R}^2}$, with the property that $T(T(v)) = v$ for every v in $\mathbb{R}^2$. Use T to find a matrix A such that $A \neq I$ and $A \neq -I$ but $A^2 = I$. (Recall that $I_{\mathbb{R}^2}$ denotes the identity transformation on $\mathbb{R}^2$, so $I_{\mathbb{R}^2}(v) = v$ and $-I_{\mathbb{R}^2}(v) = -v$. Your function T should be different from either of these.)

(6.) Suppose that n teams are playing in a large and somewhat random tournament, in which no two teams play each other more than once, but some pairs of teams may not play each other at all. You would like to have some way of comparing two teams who did not play each other.

Let A be the $n \times n$ matrix where entry A_{ij} is 1 if teams i and j play each other, and 0 if they do not. Prove that $(A^2)_{ij} = 0$ if and only if there is no team that played both team i and team j during the tournament. You might want to remember that $(A^2)_{ij} = A_{i1}A_{1j} + A_{i2}A_{2j} + \cdots + A_{1n}A_{nj}$, and ask yourself when $A_{i2}A_{2j}$ is equal to 0, and when it is equal to 1.

How do you interpret $(A^2)_{ij}$ in the case where $(A^2)_{ij} \neq 0$?

Now define a new $n \times n$ matrix B where A_{ij} is 1 if team i defeated team j, and 0 if not (either teams i and j did not play each other, or team j defeated team i). Assume there are no ties. If $(A^2)_{ij} = 6$, $(B^2)_{ij} = 4$, and $(B^2)_{ji} = 1$, what (if anything) can you conclude about how teams i and j performed against teams they both played?