Lemma

Let A be an $n \times n$ matrix with $n \geq 2$.
Suppose that i-th row of A is $(0, \ldots, 0, \underset{k}{1}, 0, \ldots, 0)$.
Then $\det(A) = (-1)^{i+k} \det(\tilde{A}_{i,k})$.

Proof.

The statement is easily proved if $n = 2$ or if $i = 1$. We will assume $n > 2$, $i > 1$.
Let C_{ij} denote the $(n-2) \times (n-2)$ matrix obtained from A by removing rows 1 and i and columns j and k. $(j \neq k)$

$$
C_{ij} = \begin{array}{cl} \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & & a_{2j} & & a_{2k} & & a_{2n} \\ \vdots & & & & & & \\ a_{i1} \; 0 & & 0 & & 1 & & 0 \\ \vdots & & & & & & \\ a_{n1} & & a_{nj} & & a_{nk} & & a_{nn} \end{pmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow i\text{-th row} \\ \\ \\ \end{array} \end{array}
$$

Then, assuming that Lemma holds true for $(n-1) \times (n-1)$ matri[ces]

$$
\det(A) = a_{11} \det \tilde{A}_{11} + \cdots + (-1)^{k} a_{1,k-1} \det \tilde{A}_{1,k-1} + (-1)^{k+1} a_{1,k} \det \tilde{A}_{1,k} + (-1)^{k+2} a_{1,k+1} \det \tilde{A}_{1,k+1} + \cdots + (-1)^{n+1} a_{1,n} \det \ldots
$$

$$
= a_{11} (-1)^{i-1+k-1} \det(C_{11}) + \cdots + (-1)^{k} a_{1,k-1} (-1)^{i-1+k-1} \det(C_{1,k-1}) + 0 +
$$

$$
+ (-1)^{k+2} a_{1,k+1} (-1)^{i-1+k} \det(C_{1,k+1}) + \cdots + (-1)^{n+1} a_{1,n} \det(C_{1,n}) \cdot (-1)^{\ldots}
$$

$$
= (-1)^{i+k} \det(\tilde{A}_{i,k}).
$$