Linear Transformations and Matrices

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Linear Transformations

- Let V and W be vector spaces over a field F.
- Let $T : V \to W$ be a function.
- We say that T is **a linear transformation** from V to W if, for all $x, y \in V$ and $c \in F$, we have
  1. $T(x + y) = T(x) + T(y)$.
  2. $T(cx) = cT(x)$.

Example

- Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

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- T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

- If $T : V \to W$,$S : V \to W$ are linear , then $T + S$ is linear.

- If V, W, and Z are vector spaces, $T : V \to W$, $S : W \to Z$ are linear, then so is $S \circ T$.

More Examples

- Let $V = \mathbb{R}^2$, $\theta \in \mathbb{R}$, and let $T : V \to V$ be defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(**rotation by the angle** θ).

- Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

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- The linear transformation $1_V : V \to V$ defined by $1_V(x) = x$ is called the **identity transformation**.

- The linear transformation $T_0 : V \to W$ defined by $T_0(x) = 0$ for all x in V is called **the zero transformation**.

The Null Space and the Range of a Linear Transformation

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- The **range** (or **image**) $R(T)$ of T is the subset of W consisting of all images (under T) of vectors in V.

Theorem. *Let* V *and* W *be vector spaces and* $T : V \longrightarrow W$ *be linear. Then* $N(T)$ *and* $R(T)$ *are subspaces of* V *and* W*, respectively.*