Basis and Dimension (cont'd)

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Lecture 7

Bases and Dimension

- A **basis** β for a vector space V is a linearly independent subset of V that generates V. If β is a bass for V, we also say that the vectors of β form a basis for V.

Theorem. *Let V be a vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V. Then $\beta$is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \dots, a_n$.

Theorem. *If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.*

Theorem. [Replacement Theorem] *Let Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \bigcup H$ generates V.*

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Corollary. *Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.*

Finite-dimensional Vector Spaces

- A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors.

- The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by $\dim(V)$.

- A vector space that is not finite-dimensional is called **infinite-dimensional**.