Gram-Schmidt Orthogonalization Process

Lecture 25

March 7, 2007

Definition

Let V be an inner product space.

1. Two vectors x and y in V are **orthogonal** if $\langle x, y \rangle = 0$.

2. A subset S of V is **orthogonal** if any two distinct vectors in S are orthogonal.

3. A vector x in V is a **unit vector** if $\|x\| = 1$.

4. A subset S of V is **orthonormal** if S is orthogonal and consists entirely of unit vectors.

5. A subset S of V is an **orthonormal basis** for V if it is an ordered basis that is orthonormal.

Fact

- If S is an orthogonal subset of V consisting of nonzero vectors then S is linearly independent.

- Any finite dimensional inner product space has an orthonormal basis.

- If $\beta = \{v_1, v_2, \ldots, v_n\}$ is an orthonormal basis for V, then for any $x \in V$

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$

The coefficients $\langle x, v_i \rangle$ are called the Fourier coefficients.

Theorem

Let V be an inner product space and $S = \{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of V. Define $S^{'} = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n.$$

Fact

Mathematical induction is used to prove that every statement in an infinite sequence of statements is true. It is done by

- *proving that the first statement in the infinite sequence of statements is true, and then*
- *proving that if any one statement in the infinite sequence of statements is true, then so is the next one.*

Definition

The simplest and most common form of mathematical induction proves that a statement holds for all natural numbers n and consists of two steps:

1. The basis: showing that the statement holds when $n = 0$.

2. The inductive step: showing that if the statement holds for $n = m$, then the same statement also holds for $n = m + 1$.

Example

Show that
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Theorem

Let V be an inner product space and $S = \{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \text{ for } 2 \leq k \leq n.$$