Inner Products and Norms

Lecture 24

March 5, 2007

Definition

Let V be a vector space over F. An **inner product** on V is a function that assigns to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle x, y\rangle$, such that for all x, y and z in V and all c in F, the following hold:

1. $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$.
2. $\langle cx, y\rangle = c\langle x, y\rangle$.
3. $\overline{\langle x, y\rangle} = \langle y, x\rangle$.
4. $\langle x, x\rangle > 0$ if $x \neq 0$.

Properties of the Inner Product

Theorem

Let V be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true:

1. $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.
2. $\langle x, cy\rangle = \overline{c}\langle x, y\rangle$.
3. $\langle x, 0\rangle = \langle 0, x\rangle = 0$.
4. $\langle x, x\rangle = 0$ *if and only if* $x = 0$.
5. *If* $\langle x, y\rangle = \langle x, z\rangle$ *for all* $x \in V$, *then* $y = z$.

The Length of a Vector

Definition

Let V be an inner product space. For $x \in V$, we define the **norm** or **length** of x by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Properties of the Norm

Theorem

Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following statements are true.

1. $\|cx\| = |c| \cdot \|x\|$.
2. $\|x\| = 0$ *if and only if* $x = 0$. *In any case,* $\|x\| \geq 0$.
3. *(Cauchy-Scwarz Inequality)* $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$.
4. *(Triangle Inequality)* $\|x + y\| \leq \|x\| + \|y\|$.

Orthogonal Vectors

Definition

Let V be an inner product space.

1. Two vectors x and y in V are **orthogonal** if $\langle x, y \rangle = 0$.
2. A subset S of V is **orthogonal** if any two distinct vectors in S are orthogonal.
3. A vector x in V is a **unit vector** if $\|x\| = 1$.
4. A subset S of V is **orthonormal** if S is orthogonal and consists entirely of unit vectors.
5. A subset S of V is an **orthonormal basis** for V if it is an ordered basis that is orthonormal.

Orthogonal Sets

Theorem

Let V be an inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V consisting on nonzero vectors. If $y \in span(S)$, then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$

Orthogonal Sets

Corollary

If S is an orthonormal set and $y \in span(S)$, then

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i.$$

Corollary

Let V be an inner product space, and let S be an orthogonal subset of V consisting of nonzero vectors. Then S is linearly independent.