Diagonalizability

Lecture 23

March 2, 2007

Linearly Independent Eigenvectors

Theorem

Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \ldots, v_k$ are eigenvectors of T such that λ_i corresponds to v_i $(1 \leq i \leq k)$, then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

n-Distinct Eigenvalues

Corollary

Let T be a linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

Characteristic Polynomials Split

Definition

A polynomial $f(t)$ in $P(F)$ **splits over** F if there are scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in F such that

$$f(t) = c(t - a_1)(t - a_2)\ldots(t - a_n).$$

Characteristic Polynomials Split

Theorem

The characteristic polynomial of any diagonalizable linear operator splits:

$$f(t) = (-1)^n (t - \lambda_1)(t - \lambda_2) \ldots (t - \lambda_n).$$

Algebraic Multiplicity

Definition

Let λbe an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The **algebraic multiplicity** of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.

Eigenspace

Definition

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define

$$E_\lambda = N(T - \lambda I).$$

The set E_λ is called the **eigenspace** of T corresponding to the eigenvalue λ.

The Dimension of the Eigenspace

Theorem

Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leq dim(E_\lambda) \leq m$.

When is T diagonalizable

Theorem

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1. *T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i.*

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Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T. Then

1. *T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda_i})$ for all i.*
2. *If T is diagonalizable and β_i is an ordered basis for E_{λ_i} for each i, then $\beta = \beta_1 \bigcup \beta_2 \bigcup \ldots \beta_k$ is an ordered basis for V consisting of eigenvectors of T.*