Eigenvalues and Eigenvectors

Lecture 22

February 28, 2007

Diagonalizable Linear Operators

Definitions

- A linear operator on a finite-dimensional vector space V is called **diagonalizable** if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.
- A square matrix A is called **diagonalizable** if L_A is diagonalizable.

Eigenvalues and Eigenvectors

Definitions

- Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an **eigenvector** of T if there exists a scalar λ such that $T(v) = \lambda v$.
- The scalar λ is called the **eigenvalue** corresponding to the eigenvector v.
- Let A be a square matrix. A nonzero vector $v \in F^n$ is called an **eigenvector** of A if v is an eigenvector of L_A: $Av = \lambda v$.
- The scalar $\lambda$is called the **eigenvalue** of A corresponding to the eigenvector v.

Theorem

A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leq j \leq n$.

Diagonalization

Definition

To **diagonalize** a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

How to Determine Eigenvalues and Eigenvectors?

Theorem

Let $A \in M_{n \times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if $det(A - \lambda I_n) = 0$.

The Characteristic Polynomial

Definition

- Let $A \in M_{n \times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the **characteristic polynomial** of A.
- Let T be a linear operator on an n-dimensional vector space V with ordered basis β. The **characteristic polynomial** $f(t)$ of T to be the characteristic polynomial of $A = [T]_\beta$.

The Characteristic Polynomial

Theorem

Let $A \in M_{n\times n}(F)$.

- *The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.*
- *A has at most n distinct eigenvalues.*

Theorem

Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.