Determinants

Lecture 21

February 26, 2007

Determinants of order 2

Definition

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix with entries from a field F, then the **determinant** of A, denoted $\det(A)$ or $|A|$, to be

$$|A| = ad - bc.$$

Theorem

The function $det : M_{2\times 2}(F) \to F$ *is a linear function of each row of a* 2×2 *matrix when the other row is fixed. That is*

$$det\begin{pmatrix} u+kv \\ w \end{pmatrix} = det\begin{pmatrix} u \\ w \end{pmatrix} + k det\begin{pmatrix} v \\ w \end{pmatrix}$$

and

$$det\begin{pmatrix} w \\ u+kv \end{pmatrix} = det\begin{pmatrix} w \\ u \end{pmatrix} + k det\begin{pmatrix} w \\ u \end{pmatrix}$$

Theorem

Suppose A is a 2×2 matrix. Then the determinant of A is nonzero if and only if A is invertible. Moreover, if A is invertible, then

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Definition

Let $\beta = \{u, v\}$ be an ordered basis for $\mathbb{R}^2$. The **orientation** of β is

$$O\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}.$$

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Note that a coordinate system $\{u, v\}$ is right handed if and only if $O\begin{pmatrix} u \\ v \end{pmatrix} = 1$.

Determinants of order 3 and higher

Definition

Suppose that A is a 3×3 matrix. Then the determinant of A is defined to be:

$$|A| = |A_{11}| - |A_{12}| + |A_{13}|.$$

This formula is called the **cofactor expansion along the first row of A.**

Determinants and Geometry (part 2)

Fact

Recall

- *The inner product between two vectors* (a, b, c) *and* (c, d, e) *in* $\mathbb{R}^3$ *is equal to the determinant*

$$\left| \begin{pmatrix} i & j & k \\ a & b & c \\ d & e & f \end{pmatrix} \right| .$$

- *The area of the parallelogram determined by two vectors is given by the absolute value of the inner product between vectors.*

Theorem

If an $n \times n$ matrix A has a row consisting entirely of zeros, then $|A| = 0$.

Properties of the Determinants

Theorem

If an $n \times n$ matrix A has two identical rows, then $|A| = 0$.

Properties of the Determinants

Theorem

If an $n \times n$ matrix A has rank less than n, then $|A| = 0$.

Elementary Operations and Determinants

Fact

- *If B is a matrix obtained by interchanging any two rows of A, then $|B| = -|A|$.*
- *If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then $|B| = k|A|$.*
- *If B is a matrix obtained by adding a multiple of one row of A to another row of A, then $|B| = |A|$.*