The Rank of a Matrix (cont'd)

Lecture 17

February 16, 2007

Definition

Let $A \in \mathrm{M}_{m \times n}(F)$. The rank of A, denoted $\mathrm{rank}(A)$ is defined to be the rank of the linear transformation $L_A : F^n \to F^m$.

How to Compute the Rank of a Matrix?

Theorem

Let A be an $m \times n$ matrix of rank r. Then $r \leq n$ and $r \leq m$, and, by means of a finite number of elementary row and columns operations, A can be transformed into the matrix

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix},$$

where $0_1, 0_2$, and 0_3 are zero matrices.

Consequences

Corollary

Let A be an $m \times n$ matrix of rank r. Then there exist invertible matrices B and C of sizes $m \times m$ and $n \times n$, respectively, such that $D = BAC$, where

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}.$$

Consequences

Corollary

Let A be an $m \times n$ matrix. Then

1. $rank(A^t) = rank(A)$.
2. *The rank of any matrix equals the maximum number of its linearly independent rows.*
3. *The rows and columns of any matrix generate subspaces of the same dimension, equal to the rank of the matrix.*

Consequences

Corollary

Every invertible matrix is a product of elementary matrices.

Rank of the Product

Theorem

Let $T : V \to W$ and $U : W \to Z$ be linear transformations on finite-dimensional vector spaces V, W, and Z, and let A and B be matrices such that the product AB is defined. Then

1. $rank(UT) \leq rank(U)$.
2. $rank(UT) \leq rank(T)$.
3. $rank(AB) \leq rank(A)$.
4. $rank(AB) \leq rank(B)$.