The Rank of a Matrix

Lecture 16

February 14, 2007

The Rank of a Matrix

Definition

Let $A \in \mathrm{M}_{m\times n}(F)$. The rank of A, denoted $\mathrm{rank}(A)$ is defined to be the rank of the linear transformation $L_A : F^n \to F^m$.

Properties

Fact

- *An* $n \times n$ *matrix is invertible if and only if its rank is* n.
- *If* $T : V \to W$ *is a linear transformation and* β *and* γ *are ordered bases for* V *and* W, *then* $\mathit{rank}(T) = \mathit{rank}([T]_\beta^\gamma)$.

Rank Preserving Operations

Theorem

Let A be an $m \times n$ matrix. If P and Q are invertible $m \times m$ and $n \times n$ matrices, respectively, then

1. $rank(AQ) = rank(A) = rank(PA)$.
2. $rank(PAQ) = rank(A)$.
3. *Elementary row and column operations on a matrix are rank-preserving.*

How to Compute the Rank of a Matrix?

Theorem

The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.

How to Compute the Rank of a Matrix?

Theorem

Let A be an $m \times n$ matrix of rank r. Then $r \leq n$ and $r \leq m$, and, by means of a finite number of elementary row and columns operations, A can be transformed into the matrix

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix},$$

where $0_1, 0_2$, and 0_3 are zero matrices.

Consequences

Corollary

Let A be an $m \times n$ matrix of rank r. Then there exist invertible matrices B and C of sizes $m \times m$ and $n \times n$, respectively, such that $D = BAC$, where

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}.$$

Consequences

Corollary

Let A be an $m \times n$ matrix. Then

1. $rank(A^t) = rank(A)$.
2. *The rank of any matrix equals the maximum number of its linearly independent rows.*
3. *The rows and columns of any matrix generate subspaces of the same dimension, equal to the rank of the matrix.*

Consequences

Corollary

Every invertible matrix is a product of elementary matrices.