

# The Rank of a Matrix

Lecture 16

February 14, 2007

# The Rank of a Matrix

## Definition

Let  $A \in M_{m \times n}(F)$ . The **rank** of  $A$ , denoted  $\text{rank}(A)$  is defined to be the rank of the linear transformation  $L_A : F^n \rightarrow F^m$ .

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- If  $T : V \rightarrow W$  is a linear transformation and  $\beta$  and  $\gamma$  are ordered bases for  $V$  and  $W$ , then  $\text{rank}(T) = \text{rank}([T]_{\beta}^{\gamma})$ .

## Theorem

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- 1  $\text{rank}(AQ) = \text{rank}(A) = \text{rank}(PA)$ .
- 2  $\text{rank}(PAQ) = \text{rank}(A)$ .
- 3 *Elementary row and column operations on a matrix are rank-preserving.*

# How to Compute the Rank of a Matrix?

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*The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.*

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## Theorem

Let  $A$  be an  $m \times n$  matrix of rank  $r$ . Then  $r \leq n$  and  $r \leq m$ , and, by means of a finite number of elementary row and columns operations,  $A$  can be transformed into the matrix

$$D = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix},$$

where  $0_1, 0_2$ , and  $0_3$  are zero matrices.

## Corollary

*Let  $A$  be an  $m \times n$  matrix of rank  $r$ . Then there exist invertible matrices  $B$  and  $C$  of sizes  $m \times m$  and  $n \times n$ , respectively, such that  $D = BAC$ , where*

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- 2 *The rank of any matrix equals the maximum number of its linearly independent rows.*

## Corollary

Let  $A$  be an  $m \times n$  matrix. Then

- 1  $\text{rank}(A^t) = \text{rank}(A)$ .
- 2 The rank of any matrix equals the maximum number of its linearly independent rows.
- 3 The rows and columns of any matrix generate subspaces of the same dimension, equal to the rank of the matrix.



## Corollary

*Every invertible matrix is a product of elementary matrices.*