Invertibility and Isomorphisms

February 5, 2007

Invertibility of Linear Maps

- Let V and W be vector spaces, and let $T : V \to W$ be linear.
- A function $U : W \to V$ is an **inverse** of T is $TU = 1_W$ and $UT = 1_V$.
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- Recall that the inverse of a function is unique and is denoted by T^{-1}.

Properties of the Inverse of A Linear Map

- $(TU)^{-1} = U^{-1}T^{-1}$.
- $(T^{-1})^{-1} = T$
- A map is invertible if and only if it is one-to-one and onto.
- A linear map $T : V \to W$ is invertible if and only if $\text{rank}(T) = \dim(V)$.

The Inverse of A Linear Map is Linear

Theorem. *Let $T : V \to W$ be linear, where V and W are vector spaces. Suppose that T is invertible. Then T^{-1} is linear.*

The Inverse of a Matrix

- Let A be an $n \times n$ matrix.
- We say that A is **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I.$$

The Matrix of An Invertible Linear Map is Invertible

Theorem. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $T : V \longrightarrow W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Moreover, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.*

Isomorphisms

- Let V and W be vector spaces, and let $T : V \to W$ be a linear map.
- If T is invertible, we call it an **isomorphism**.
- We say that V and W are **isomorphic**.

Theorem. *Let V and W be two vector spaces. Suppose that V is finite-dimensional. Then V is isomorphic t oW if and only if W is finite dimensional and dim$(V) =$ dim(W).*

All Finite Dimensional Spaces of a Fixed Dimension are Isomorphic

Corollary. *Let V be a vector space over F. Then V is isomorphic to F^n if and only if dim$(V) = n$. Moreover an isomorphism is given by the function $\phi_\beta : V \longrightarrow F^n$ defined by*

$$\phi_\beta(x) = [x]_\beta,$$

where β is an ordered basis for V.

Linear Maps "are" Matrices

Theorem. *Let V and W be finite-dimensional vector spaces over F of dimensions n and m, and let β and γ be ordered bases for V and W, respectively. Then the function $\phi : \mathcal{L}(V, W) \to M_{m\times n}(F)$, defined by*

$$\phi(T) = [T]_\beta^\gamma$$

is an isomorphism.