The Matrix Representation of a Linear Transformation

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- Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vector space V.

- For a vector v in V write it as a linear combination of the vectors in the basis:

$$v = \sum_{i=1}^{n} a_i u_i.$$

- The **coordinate vector of x relative to β**, denoted $[x]_\beta$ is

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

- Let $T : V \to W$ be linear.

- Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ be basis for V and, respectively, W.

- Then we can write

$$T(v_j) = \sum_{i=1}^{n} a_{ij} w_i \text{ for } 1 \leq j \leq n.$$

- We call the $m \times n$ matrix A defined by the scalars a_{ij} the **matrix representation of T in the ordered bases β and γ**; we write $A = [T]_\beta^\gamma$.

Examples:

- Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(a_1, a_2, a_3) = (3a_1 + a_2, a_1 + a_3, a_1 - a_3)$.

- Consider the standard ordered basis $\{e_1, e_2, e_3\}$. With respect to this basis the coordinate vector of an element (a_1, a_2, a_3) is

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

- The matrix representation of T is

$$\begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

- Suppose now that we choose as a basis for $V = \mathbb{R}^3$ the set $\beta = \{(1,0,0),(1,1,0),(1,1,1)\}$.

- Then the coordinate vector of an element (a_1, a_2, a_3) is

$$\begin{pmatrix} a_1 - a_2 \\ a_2 - a_3 \\ a_3 \end{pmatrix}.$$

- The matrix representation of T with respect to this basis is

$$\begin{pmatrix} 3 & 4 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

One more example:

- Consider the linear transformation $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be defined by $T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$.

- What is the matrix representation with respect to the standard basis in $P_2(\mathbb{R})$ and $\mathbb{R}^3$?

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- What is the matrix representation with respect to the standard basis in $P_2(\mathbb{R})$ and $\mathbb{R}^3$?

- What if we choose a basis in $P_2(\mathbb{R})$ having elements with the same degree?

Theorem. *Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $T : V \to W$ be linear. Then, for each $u \in V$, we have*

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta.$$

The Matrix Representation of the Sum of two Linear Transformations

Theorem. *Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U : V \to W$ be linear transformations. Then*

1. $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ *and*

2. $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ *for all scalars* a.

The space of Linear Transformations

- Let V and W be vector spaces over the field F.
- We denote by $\mathcal{L}(V, W)$ the set of all linear transformations from V into W.
- If $V = W$ we write shortly $\mathcal{L}(V)$.

Composition of Linear Transformations and Matrix Multiplication

- **Recall:** The composition of two linear maps is linear.

- That is, if $T : V \to W$ and $S : W \to Z$ are linear, where V, W, and Z are vector spaces over the same field, then $ST = S \circ T : V \to Z$ is linear.

Matrix Multiplication

- Let A be an $m \times n$ matrix and B be an $n \times p$ matrix.
- The product of A and B, denoted AB is the $n \times p$ matrix such that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \text{ for } 1 \leq i \leq m, 1 \leq j \leq p.$$

- Matrix multiplication satisfies all the "nice" properties: (see Theorem 2.12)
  - $A(B + C) = AB + AC$ and $(D + E)A = DA + EA$.
  - $a(AB) = (aA)B = A(aB)$.
  - $I_m A = A = AI_n$ if A is an $m \times n$-matrix.

Theorem. *Let* V, W, *and* Z *be finite-dimensional vector spaces with ordered bases* α, β, *and* γ *respectively. Let* $T : V \to W$ *and* $U : W \to Z$ *be linear transformations. Then*

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$