Worksheet for May 28

MATH 24 — SPRING 2014

Sample Solutions

Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and let A also denote the corresponding left multiplication transformation $\mathbb{R}^5 \to \mathbb{R}^5$. The characteristic polynomial of A splits $\det(A - tI) = -t^3(1-t)^2$, but it is not diagonalizable.

(A) Note that

$$A^2 = \begin{pmatrix} 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

1.– Explain why the generalized eigenspace K_0 equals the null space of A^2.

Solution — We know from Theorem 7.4(c) that $\dim(\mathsf{K}_0)$ equals the algebraic multiplicity of the eigenvalue 0, which is 3. Since the null space of $A^2 = (A - 0I)^2$ has dimension 3, it must equal K_0.

2.– Explain why the dot diagram corresponding to the eigenvalue 0 must be: $\begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array}$

Solution — The number of dots in the first row is the nullity of A, which is 2. We must have 3 dots in total, so the pattern must be : $\begin{array}{cc} \bullet & \bullet \\ \bullet & \end{array}$

3.– Find a vector $x_2 \in \mathsf{K}_0$ such that $x_1 = Ax_2 \neq 0$.

Solution — A basis for the null space of A^2 is $\{e_1, e_2, e_3\}$. One of these vectors must be outside the null space of A. By inspection, $x_2 = e_3$ works since

$$x_1 = Ae_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

4.– Find a vector $x_3 \in \mathsf{K}_0$ such that $\{x_1, x_3\}$ is a basis for the null space of A.

Solution — A basis for the null space of A is $\{e_1, e_2\}$. By the Replacement Theorem, one of these two basis vectors must form a basis for the null space of A along with the vector x_1 above. Since neither e_1 nor e_2 is a multiple of x_1, they actually both work. So we can pick $x_3 = e_1$.

(B) Note that

$$(A-I)^2 = A^2 - 2A + I = \begin{pmatrix} 1 & 0 & -2 & -1 & 0 \\ 0 & 1 & -4 & 1 & 3 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1.– Explain why the generalized eigenspace K_1 equals the null space of $(A-I)^2$.

Solution — We know from Theorem 7.4(c) that $\dim(\mathsf{K}_1)$ equals the algebraic multiplicity of the eigenvalue 1, which is 2. Since the null space of $(A-I)^2$ has dimension 2, it must equal K_0.

2.– Explain why the dot diagram corresponding to the eigenvalue 1 must be: $\begin{array}{c}\bullet\\ \bullet\end{array}$

Solution — The number of dots in the first row is the nullity of $A - I$, which is 1. We must have 2 dots in total, so the pattern must be : $\begin{array}{c}\bullet\\ \bullet\end{array}$

3.– Find a vector $y_2 \in \mathsf{K}_1$ such that $y_1 = (A-I)y_2 \neq 0$.

Solution — A basis for the null space of $(A-I)^2$ is

$$\left\{ \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

One of these two vectors must be outside the null space of $A - I$, since it has dimension 1. By inspection, we can pick

$$y_2 = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{since} \quad y_1 = (A-I)y_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

4.– Check that y_1 generates the null space of $A - I$.

Solution — The null space of $A - I$ is one dimensional and it contains the nonzero vector y_1, so the null space of $A - I$ must be $\operatorname{span}\{y_1\}$.

(C) Verify that $\beta = \{x_1, x_2, x_3, y_1, y_2\}$ is a basis for $\mathbb{R}^5$ and compute the matrix representation $[L_A]_\beta$.

Solution — We could check that the vectors we found are linearly independent, or rely on Theorem 7.4(b) to see that β is a basis for $\mathbb{R}^5$.

The way we picked the vectors in β leads to the equations:

$$\begin{aligned} Ax_1 &= 0, & [Ax_1]_\beta &= (0,0,0,0,0); \\ Ax_2 &= x_1, & [Ax_2]_\beta &= (1,0,0,0,0); \\ Ax_3 &= 0, & [Ax_3]_\beta &= (0,0,0,0,0); \end{aligned}$$

and

$$\begin{aligned} (A-I)y_1 &= 0, & Ay_1 = y_1, [Ay_1]_\beta &= (0,0,0,1,0); \\ (A-I)y_2 &= y, & Ay_2 = y_1 + y_2, [Ay_2]_\beta &= (0,0,0,1,1). \end{aligned}$$

Therefore,

$$[L_A]_\beta = \begin{pmatrix} \mathbf{0} & \mathbf{1} & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

which is exactly the Jordan canonical form the dot patterns we found predicted.