Worksheet for May 1

MATH 24 — SPRING 2014

Sample Solutions

(A) Let $A = \begin{pmatrix} -1 & -4 & 1 \\ 2 & 2 & 0 \\ -3 & 0 & 2 \end{pmatrix}$.

1.– Compute the determinant of A using cofactor expansion.

Solution — We can do cofactor expansion along any row or column; those with some zeros save you some work.

Along the second row we have:

$$\begin{aligned}\sum_{j=1}^{3}(-1)^{2+j}A_{2j}\det(\widetilde{A}_{2j}) &= -2\det\begin{pmatrix} -4 & 1 \\ 0 & 2 \end{pmatrix} + 2\det\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix} \\ &= -2((-4)(2)-(0)(1)) + 2((-1)(2)-(-3)(1)) = 18.\end{aligned}$$

2.– Using only type 1 and type 3 row operations, make A into an upper triangular matrix U.

Solution — We can manage only with type 3 row operations. After adding 2 times the first row to the second and then adding -3 times the first row to the third, we obtain:

$$\begin{pmatrix} -1 & -4 & 1 \\ 0 & -6 & 0 \\ 0 & 12 & -1 \end{pmatrix}.$$

Finally, adding -2 times the second row to the third, we obtain the upper triangular matrix:

$$U = \begin{pmatrix} -1 & -4 & 1 \\ 0 & -6 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

3.– Verify that $\det(A) = (-1)^n \det(U)$ where n is the number of type 1 row operations you performed in part 2.

Solution — Indeed, $18 = (-1)(-6)(3)$.

(B) Given a $n \times n$ matrix A, let C be the matrix of cofactors of A. That is $C_{ij} = (-1)^{i+j}\det(\widetilde{A}_{ij})$, where $\widetilde{A}_{ij}$ is obtained from A by deleting the i-th row and the j-th column.

1.– Explain why the (i, j)-th entry of AC^t is

$$\sum_{k=1}^{n} (-1)^{j+k} A_{ik} \det(\widetilde{A}_{jk}).$$

Solution — In general, the (i, j)-th entry of AB is

$$\sum_{k=1}^{n} A_{ik} B_{kj}.$$

If $B = C^t$, then the (k, j)-th entry of B is the (j, k)-th entry of C, that is

$$B_{kj} = (-1)^{j+k} \det(\widetilde{A}_{jk}).$$

The desired formula follows immediately.

2.– Explain why

$$\sum_{k=1}^{n} (-1)^{j+k} A_{ik} \det(\widetilde{A}_{jk}) = 0$$

when $i \neq j$. (*Hint*: Suppose, for the purpose of a thought experiment, that the j-th row of A is identical to the i-th row of A. What would be the cofactor expansion of $\det(A)$ along the j-th row?)

Solution — If the i-th and j-th rows of A are indeed identical, then by cofactor expansion along the j-th row

$$\begin{aligned} \det(A) &= \sum_{k=1}^{n} (-1)^{j+k} A_{jk} \det(\widetilde{A}_{jk}) \\ &= \sum_{k=1}^{n} (-1)^{j+k} A_{ik} \det(\widetilde{A}_{jk}). \end{aligned}$$

But $\det(A) = 0$ since A has two identical rows.

3.– Explain why $AC^t = \det(A)I$ where I is the $n \times n$ identity matrix.

Solution — The previous parts tell us that the off-diagonal entries of AC^t are all zeros. For the (i, i)-th entry, we have

$$\sum_{k=1}^{n} (-1)^{i+k} A_{ik} \det(\widetilde{A}_{ik}),$$

which is the cofactor expansion along the i-th row of A. So all the diagonal entries equal $\det(A)$.

4.– Conclude that if $\det(A) \neq 0$ then A is invertible and $A^{-1} = \dfrac{1}{\det(A)} C^t$.

Solution — The last part tells us that if $\det(A) \neq 0$ then

$$A\left(\frac{1}{\det(A)}C^t\right) = \frac{1}{\det(A)}(AC^t) = \frac{1}{\det(A)}\det(A)I = I.$$

Since A is a square matrix with a right inverse, it is invertible and A^{-1} is the right inverse $\frac{1}{\det(A)}C^t$.