Worksheet for April 28

MATH 24 — SPRING 2014

Sample Solutions

As a warm up, let's prove a useful fact about matrix multiplication.

THEOREM. *Suppose A is a $p \times q$-matrix and B is a $q \times r$ matrix, over the same field F.*

(a) *If the columns of B are $v_1, v_2, \dots, v_r$ then the columns of AB are $Av_1, Av_2, \dots, Av_r$.*

(b) *Every linear dependency between the columns of B also holds between the same columns of AB.*

Proof. For (a), recall that the i-th column of AB is simply $(AB)e_i$. Because matrix multiplication is associative, we have $(AB)e_i = A(Be_i) = Av_i$ since Be_i is the i-th column of B.

For (b), suppose that $c_1, c_2, \dots, c_r$ are scalars such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_r v_r = 0.$$

Then, using distributivity of matrix multiplication and the fact that scalar multiplication commutes with matrix multiplication:

$$\begin{aligned} 0 = A(c_1 v_1 + c_2 v_2 + \cdots + c_r v_r) &= A(c_1 v_1) + A(c_2 v_2) + \cdots + A(c_r v_r) \\ &= c_1(Av_1) + c_2(Av_2) + \cdots + c_r(Av_r). \end{aligned}$$

Therefore, the exact same linear dependency holds between the columns of AB. □

Now let A be the 4×6 matrix

$$\begin{pmatrix} 4 & -1 & 5 & 7 & 4 & -1 \\ 0 & 2 & -2 & 2 & -5 & 7 \\ 3 & -4 & 7 & 2 & 2 & -3 \\ 1 & 6 & -5 & 8 & -3 & 10 \end{pmatrix}$$

over the field $\mathbb{C}$. Suppose after some elementary row operations starting from A, you obtained the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1.– Show that there is an invertible matrix Q such that

$$QA = \begin{pmatrix} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution — After Theorem 3.1, the matrix Q is the product of the elementary matrices corresponding to the elementary row operations performed on A to obtain the new matrix.

By Theorem 3.2, elementary matrices are invertible. By Exercise 4 of Section 2.4, products of invertible matrices are invertible. It follows that Q is invertible.

2.– Show that the third and fourth columns of A are linear combinations of the first two columns of A.

Solution — This is clear for QA where the first two columns are e_1 and e_2, respectively, and the third and fourth columns of QA are $e_1 - e_2$ and $2e_1 + e_2$, respectively.

By part (b) of the Theorem, the exact same relations must hold between the corresponding columns of $A = Q^{-1}(QA)$.

3.– Show that the fifth column of A is not a linear combination of the first two columns of A.

Solution — This is clear for QA where the first two columns are e_1 and e_2, respectively, and the fifth column is e_3 — three linearly independent vectors.

If there were a nontrivial linear dependency between the first, second and fifth columns of A, then the same would hold between the same columns of QA by part (b) of the Theorem. Therefore, the first, second and fifth columns of A must be linearly independent.

4.– Show that the first, second and fifth columns of A form a basis for the subspace of $\mathbb{R}^4$ generated by the columns of A.

Solution — We have just observed that they are linearly independent. Since $\operatorname{rank}(A) = 3$, they must form a basis for $\mathsf{R}(L_A)$, subspace of $\mathbb{R}^4$ generated by the columns of A.

5.– Suppose that P is an invertible matrix such that the first two columns of PA are e_1 and e_2, respectively. Show that the third and fourth columns of PA must then be $e_1 - e_2$ and $2e_1 + e_2$, respectively.

Solution — Let v_1, v_2, v_3, v_4 denote the first four columns of A. We have observed in part 2 that $v_3 = v_1 - v_2$ and $v_4 = 2v_1 + v_2$.

By part (a) of the Theorem, the first four columns of PA are Pv_1, Pv_2, Pv_3, Pv_4. Therefore,

$$Pv_3 = P(v_1 - v_2) = Pv_1 - Pv_2 = e_1 - e_2$$

and

$$Pv_4 = P(2v_1 + v_2) = 2Pv_1 + Pv_2 = 2e_1 + e_2.$$