Worksheet for April 24

MATH 24 — SPRING 2014

Sample Solutions

Consider the system of linear equations

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5, \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9, \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15. \end{aligned}$$

1.– Let A be the matrix of coefficients of this system, form the augmented matrix $\left(A \mid I_3 \right)$, and convert it into row echelon form using a sequence of elementary row operations.

Solution — There are many ways to achieve this. Here is one sequence of row operations that works:

$$\left(\begin{array}{ccccc|ccc} 0 & 3 & -6 & 6 & 4 & 1 & 0 & 0 \\ 3 & -7 & 8 & -5 & 8 & 0 & 1 & 0 \\ 3 & -9 & 12 & -9 & 6 & 0 & 0 & 1 \end{array}\right)$$

(i) Add -1 times the third row from the second:

$$\left(\begin{array}{ccccc|ccc} 0 & 3 & -6 & 6 & 4 & 1 & 0 & 0 \\ 0 & 2 & -4 & 4 & 2 & 0 & 1 & -1 \\ 3 & -9 & 12 & -9 & 6 & 0 & 0 & 1 \end{array}\right)$$

(ii) Exchange the first and third rows:

$$\left(\begin{array}{ccccc|ccc} 3 & -9 & 12 & -9 & 6 & 0 & 0 & 1 \\ 0 & 2 & -4 & 4 & 2 & 0 & 1 & -1 \\ 0 & 3 & -6 & 6 & 4 & 1 & 0 & 0 \end{array}\right)$$

(iii) Add $-3/2$ times the second row to the third:

$$\left(\begin{array}{ccccc|ccc} 3 & -9 & 12 & -9 & 6 & 0 & 0 & 1 \\ 0 & 2 & -4 & 4 & 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -3/2 & 3/2 \end{array}\right)$$

Note that the augmented part accumulates the product of the three elementary matrices corresponding to the operations above:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3/2 & 3/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3/2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

2.– Suppose you obtained $\left(\begin{array}{c|c} B & C \end{array}\right)$ after part 1. Solve the system

$$B\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = C\begin{pmatrix} -5 \\ 9 \\ 15 \end{pmatrix}.$$

Check that your solutions are also solutions of the original system of linear equations.

Solution — From the above, we have the system:

$$\begin{aligned} 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \\ 2x_2 - 4x_3 + 4x_4 + 2x_5 &= -6 \\ x_5 &= 4. \end{aligned}$$

This system has infinitely many solutions, one of which is $s_0 = (-24, -7, 0, 0, 4)$. The other solutions are of the form $s_0 + s_h$ where s_h is in the null space of left multiplication by B.

Because C is invertible, with inverse

$$C^{-1} = \begin{pmatrix} 0 & 3/2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and $B = CA$, the two systems have the exact same solutions. Indeed, if $B(x_1, x_2, x_3, x_4, x_5) = (15, -6, 4)$ then

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = C^{-1}B\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = C^{-1}\begin{pmatrix} 15 \\ -6 \\ 4 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ 15 \end{pmatrix}.$$

Indeed, if $A(x_1, x_2, x_3, x_4, x_5) = (-5, 9, 15)$ then

$$B\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = CA\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = C\begin{pmatrix} -5 \\ 9 \\ 15 \end{pmatrix} = \begin{pmatrix} 15 \\ -6 \\ 4 \end{pmatrix}.$$

If you have multiple systems $Ax = b$ to solve, each with the same coefficient matrix but different target vectors b, this process is a very economical way to solve all of them. Indeed, Cb is easy to compute and, since B is in echelon form, the equivalent system $Bx = Cb$ is easy to solve. However, if you only have one system $Ax = b$ to solve, finding an echelon form for the augmented matrix $\left(\begin{array}{c|c} A & b \end{array}\right)$ as described in section 3.4 is not any longer.

3.– Continue from the echelon form for A you obtained in part 1 and find the *reduced* row echelon form of A using some more elementary row operations.

Solution — After some steps, we obtain that the reduced row echelon form of A is

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 9 & 26/3 \\ -1 & 2 & -2 \\ 1 & -3/2 & 3/2 \end{pmatrix} A.$$

4.– Looking at the reduced row echelon form of A you obtained in part 3, explain how you can reach the conclusion that

$$\begin{pmatrix} -6 \\ 8 \\ 12 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -7 \\ -9 \end{pmatrix}$$

and

$$\begin{pmatrix} 6 \\ -5 \\ -9 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -7 \\ -9 \end{pmatrix}.$$

Solution — From the reduced row echelon form of A, we see that the homogeneous system $Ax = 0$ is equivalent to the simple system

$$\begin{aligned} x_1 &= 2x_3 - 3x_4, \\ x_2 &= 2x_3 - 2x_4, \\ x_5 &= 0. \end{aligned}$$

Choosing $x_3 = -1, x_4 = 0$, we obtain the solution $(-2, -2, -1, 0, 0) = -2e_1 - 2e_2 - e_3$, which means that

$$-2Ae_1 - 2Ae_2 - Ae_3 = 0 \quad \text{or} \quad Ae_3 = -2Ae_1 - 2Ae_2.$$

Since Ae_1, Ae_2, Ae_3 are the first three columns of A, we obtain

$$\begin{pmatrix} -6 \\ 8 \\ 12 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -7 \\ -9 \end{pmatrix}.$$

Choosing $x_3 = 0, x_4 = -1$, we obtain the solution $(3, 2, 0, -1, 0) = 3e_1 + 2e_2 - e_4$, which means that

$$3Ae_1 + 2Ae_2 - Ae_4 = 0 \quad \text{or} \quad Ae_4 = 3Ae_1 + 2Ae_2.$$

Since Ae_1, Ae_2, Ae_3 are the first, second and fourth columns of A, we obtain

$$\begin{pmatrix} 6 \\ -5 \\ -9 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -7 \\ -9 \end{pmatrix}.$$

This process is completely general: looking at the reduced row echelon form B of a $m \times n$ matrix A, if the i-th column Be_i has no pivot then the entries of Be_i spell out a way to write the i-th column Ae_i as a linear combination of columns $Ae_1, \ldots, Ae_{i-1}$. It follows from this that the columns of A that correspond to pivots in B generate the column space $\mathrm{span}\{Ae_1, Ae_2, \ldots, Ae_n\}$. In fact, they always form a basis for the column space since the rank of A equals the number of pivots in B.