Worksheet for April 21

MATH 24 — SPRING 2014

Sample Solutions

(A) For each of the elementary row operations below, (a) find the elementary matrix E such that EA is the result of the row operation on the 3×5 matrix A, (b) find the elementary matrix $\bar{E}$ corresponding to the inverse operation so that $\bar{E}EA$ results in the original 3×5 matrix A, (c) check that $\bar{E}E = I_3$.

1.– Multiplying the second row by the nonzero scalar a.

2.– Exchanging the second and third rows.

3.– Adding a times the first row to the third row.

Solution — 1.– The inverse of 'multiplying the second row by the nonzero scalar a' is 'multiplying the second row by the nonzero scalar $1/a$.'

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.– Exchanging the second and third rows is its own inverse!

$$E = \bar{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

3.– The inverse of 'adding a times the first row to the third row' is 'subtracting a times the first row to the third row.'

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & 0 & 1 \end{pmatrix}.$$

(B) Let

$$A = \begin{pmatrix} 1 & -3 & 0 \\ -1 & 0 & 0 \\ 2 & -6 & 1 \end{pmatrix}$$

be a 3×3 matrix over $\mathbb{R}$.

1.– By performing a sequence of elementary row operations on A, find a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \cdots E_2 E_1 A = I_3$.

2.– Find the corresponding sequence $\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_k$ of elementary matrices representing the inverse row operations that you did in part 1 (in the order you performed them).

3.– Check that $A = \bar{E}_1 \bar{E}_2 \cdots \bar{E}_k$ where $\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_k$ are the elementary matrices you found in part 2.

4.– Check that $A^{-1} = E_k \cdots E_2 E_1$ where $E_1, E_2, \ldots, E_k$ are the elementary matrices you found in part 1.

Solution — There are many ways to do part 1. One combination that works is:

Step 1. Add 1 times the first row to the second.

Step 2. Add -2 times the first row from the third.

Step 3. Add -1 times the second row from the first.

Step 4. Multiply the second row by $-1/3$.

In this case, the four elementary matrices for part 1 are:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The four elementary matrices for part 2 are:

$$\bar{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \bar{E}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For part 3, we have

$$\begin{aligned}
\bar{E}_1 \bar{E}_2 \bar{E}_3 \bar{E}_4 &= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -3 & 0 \\ -1 & 0 & 0 \\ 2 & -6 & 1 \end{pmatrix}.
\end{aligned}$$

This is indeed the matrix A.

For part 4, we obtain

$$
\begin{aligned}
E_4E_3E_2E_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1/3 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 \\ -1/3 & -1/3 & 0 \\ -2 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

To see that this is A^{-1}, we can verify that the two matrices give the identity when multiplied together:

$$
\begin{pmatrix} 0 & -1 & 0 \\ -1/3 & -1/3 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ -1 & 0 & 0 \\ 2 & -6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & 0 \\ -1 & 0 & 0 \\ 2 & -6 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -1/3 & -1/3 & 0 \\ -2 & 0 & 1 \end{pmatrix}.
$$