Worksheet for April 17

MATH 24 — SPRING 2014

Sample Solutions

(A) Let $\alpha = \{1, x, x^2\}$ and $\beta = \left\{\frac{1}{2}x^2 - \frac{1}{2}x, 1 - x^2, \frac{1}{2}x^2 + \frac{1}{2}x\right\}$ be the two ordered bases for $\mathsf{P}_2(\mathbb{R})$ from Quiz 3.

1.– Compute the change of coordinate matrix Q from β to α.

2.– Compute the change of coordinate matrix Q^{-1} from α to β.

3.– Verify that $QQ^{-1} = I$ and $Q^{-1}Q = I$.

Solution —

1.– Looking at the coefficients of the elements of β, we see that

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{pmatrix}.$$

2.– As in Quiz 3, $[1]_\beta = (1, 1, 1)$, $[x]_\beta = (-1, 0, 1)$, $[x^2]_\beta = (1, 0, 1)$, so

$$Q^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Different bases for the same space have different properties. The basis β comes from Lagrange interpolation and it has the interesting property that

$$f(x) = f(-1)(\tfrac{1}{2}x^2 - \tfrac{1}{2}x) + f(0)(1 - x^2) + f(1)(\tfrac{1}{2}x^2 + \tfrac{1}{2}x)$$

for every $f(x) \in \mathsf{P}_2(\mathbb{R})$. The standard basis α has other interesting properties, such as making derivatives easy to compute. The change of coordinate matrices allow you to go back and forth between α and β and simultaneously exploit the nice properties of each basis.

(B) Let $(a, b) \in \mathbb{R}^2$ be such that $a^2 + b^2 = 1$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection across the line $\mathsf{L} = \operatorname{span}\{(a, b)\}$.

1.– Compute the matrix representation $[T]_\beta$ with respect to the ordered basis $\beta = \{(a, b), (b, -a)\}$. (Note that $(b, -a)$ is perpendicular to the line L.)

2.– Show that $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}^2 = I$.

3.– Compute the matrix representation $[T]_\alpha$ with respect to the standard ordered basis $\alpha = \{e_1, e_2\}$.

Solution —

1.– Since the reflection fixes the line L, $T(a,b) = (a,b)$. Since $(b,-a)$ is perpendicular to the line L, $T(b,-a) = -(b,-a) = (-b,a)$. Therefore

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2.– This is a straightforward calculation:

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ba+(-a)b \\ ab+b(-a) & b^2+(-a)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

3.– The matrix that changes β-coordinates to α-coordinates is

$$Q = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

By part 2, $Q^{-1} = Q$ is, surprisingly, the matrix that changes α-coordinates to β-coordinates. By Theorem 2.23, we have

$$\begin{aligned} [T]_\alpha &= Q[T]_\beta Q \\ &= \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= \begin{pmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{pmatrix}. \end{aligned}$$

Computing the matrix $[T]_\alpha$ directly is quite a challenging geometry problem... (Try it!) By carefully choosing an basis β tailored to T, rather than the standard basis α, the problem suddenly becomes much easier! In Chapter 6, we will discuss how to arrive at this particular choice of basis.

(C) Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix}$ and let $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ denote left multiplication by A.

1.– Find a basis for $\mathsf{N}(L_A)$ and a basis for $\mathsf{R}(L_A)$. Check that the union of the two bases you just found forms a basis β for $\mathbb{R}^3$.

2.– Compute the matrix representation $[L_A]_\beta$ with respect to the ordered basis β you just found.

3.– Show that $A^2 = 9A$ without computing A^2.

Solution —

1.– There are several possible choices, for example: $\{(2,-2,1),(2,1,-2)\}$ is a basis for $\mathsf{N}(T)$; $\{(1,2,2)\}$ is a basis for $\mathsf{R}(T)$. Indeed, $\beta = \{(1,2,2),(2,-2,1),(2,1,-2)\}$ is a basis for $\mathbb{R}^3$.

2.– Since $L_A(1,2,2) = (9,18,18) = 9(1,2,2)$ and $L_A(2,-2,1) = L_A(2,1,-2) = (0,0,0)$, we see that

$$[L_A]_\beta = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Regardless of your choices for part 1, you should get a matrix with one 9 along the diagonal and all other entries 0. (Pay attention to how having a basis β that contains a basis for $\mathsf{N}(T)$ affects the structure of $[T]_\beta^\gamma$.)

3.– It is easy to see from part 2 that $[L_A^2]_\beta = [L_A]_\beta^2 = 9[L_A]_\beta$. Since the linear transformation $T \in \mathcal{L}(\mathbb{R}^3,\mathbb{R}^3) \mapsto [T]_\beta \mathsf{M}_{3\times 3}(\mathbb{R})$ is an isomorphism by Theorem 2.20, it is one-to-one and we conclude that $L_A^2 = 9L_A$. By Theorem 2.15(c,e), we then see that $L_{A^2} = L_{9A}$. Since the linear transformation $M \in \mathsf{M}_{3\times 3}(\mathbb{R}) \mapsto L_M \in \mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$ is also an isomorphism (it is the inverse of the isomorphism above), it follows from $L_{A^2} = L_{9A}$ that $A^2 = 9A$.

This back-and-forth translation process is a very common use of isomorphisms. Our goal is a statement about the matrix A but the matrix $[L_A]_\beta$ is much easier to understand because it has much simpler structure than A. The isomorphisms between $\mathcal{L}(\mathbb{R}^3,\mathbb{R}^3)$ and $\mathsf{M}_{3\times 3}(\mathbb{R})$ allow us to translate properties of A into properties of $[L_A]_\beta$ and then back into properties of A.