Worksheet for April 11

MATH 24 — SPRING 2014

Sample Solutions

(A) Consider the ordered basis $\alpha = \{1, x+1, x^2+x+1, x^3+x^2+x+1\}$ for $\mathsf{P}_3(\mathbb{R})$. Compute the vectors $[x^2]_\alpha$, $[x^3-2x^2+1]_\alpha$ and $[(x+1)^3]_\alpha$.

Solution — Since

$$x^2 = (x^2+x+1)-(x+1),$$

we see that

$$[x^2]_\alpha = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

Since

$$x^3-2x^2+1 = (x^3+x^2+x+1)-3(x^2+x+1)+2(x+1)+(1),$$

we see that

$$[x^3-2x^2+1]_\alpha = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}.$$

Since

$$(x+1)^3 = x^3+3x^2+3x+1 = (x^3+x^2+x+1)+2(x^2+x+1)-2(1),$$

we see that

$$[(x+1)^3]_\alpha = \begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

(B) Let $E : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ be the linear transformation defined by $E(f(x)) = f(x+1)$. For example,

$$E(x^2-2x) = (x+1)^2-2(x+1) = (x^2+2x+1)-(2x+2) = x^2-1.$$

1.– Compute the matrix representation $[E]_\gamma$ with respect to the standard ordered basis $\gamma = \{1, x, x^2\}$.

2.– Compute the matrix representation $[E]_\beta^\gamma$, with respect to the bases $\beta = \{1, x+1, x^2+x+1\}$ and $\gamma = \{1, x, x^2\}$.

3.– Compute the matrix representation $[E]_\gamma^\beta$, with respect to the bases $\beta = \{1, x+1, x^2+x+1\}$ and $\gamma = \{1, x, x^2\}$.

Solution —

1.– We have:

$$\begin{array}{rcl} E(1) = 1 & \text{so} & [E(1)]_\gamma = (1,0,0); \\ E(x) = x+1 & \text{so} & [E(x)]_\gamma = (1,1,0); \\ E(x^2) = x^2+2x+1 & \text{so} & [E(x^2)]_\gamma = (1,2,1). \end{array}$$

Therefore

$$[E]_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

2.– We have:

$$\begin{array}{rcr} E(1) = 1 & \text{so} & [E(1)]_\gamma = (1,0,0); \\ E(x+1) = x+2 & \text{so} & [E(x+1)]_\gamma = (2,1,0); \\ E(x^2+x+1) = x^2+3x+3 & \text{so} & [E(x^2+x+1)]_\gamma = (3,3,1). \end{array}$$

Therefore

$$[E]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

3.– We have:

$$\begin{array}{lcl} E(1) = 1 & \text{so} & [E(1)]_\beta = (1,0,0); \\ E(x) = x+1 & \text{so} & [E(x)]_\beta = (0,1,0); \\ E(x^2) = (x^2+x+1)+(x+1)-(1) & \text{so} & [E(x^2)]_\beta = (-1,1,1). \end{array}$$

Therefore

$$[E]_\gamma^\beta = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(C) Let $L : \mathsf{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by $L(f(x)) = (f(1), f(2), f(3))$. For example,

$$L(x^2-1) = \begin{pmatrix} (1)^2-1 \\ (2)^2-1 \\ (3)^2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix}.$$

1.– Compute the matrix representation of $[L]^{\delta}_{\gamma}$ where $\delta = \{e_1, e_2, e_3\}$ (the standard ordered basis for $\mathbb{R}^3$) and $\gamma = \{1, x, x^2\}$.

2.– Use Theorem 2.11 to compute the matrix representation $[LE]^{\delta}_{\beta}$ where $\delta = \{e_1, e_2, e_3\}$, $\beta = \{1, x+1, x^2+x+1\}$, and $E : \mathsf{P}_2(\mathbb{R}) \to \mathsf{P}_2(\mathbb{R})$ is as in part (B).

3.– Use Theorem 2.14 to compute $LE(x^2 + 2x)$ and verify the result by direct computation.

Solution —

1.– Since

$$L(1) = (1,1,1), \quad L(x) = (1,2,3), \quad L(x^2) = (1,4,9),$$

we have

$$[L]^{\delta}_{\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

2.– By Theorem 2.11, $[LE]^{\delta}_{\beta} = [L]^{\delta}_{\gamma}[E]^{\gamma}_{\beta}$, so

$$[LE]^{\delta}_{\beta} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 4 & 13 \\ 1 & 5 & 21 \end{pmatrix}.$$

3.– By Theorem 2.14, $[LE(x^2+2x)]_{\delta} = [LE]^{\delta}_{\beta}[x^2+2x]_{\beta}$. Since δ is the standard basis of $\mathbb{R}^3$, we also have $LE(x^2+2x) = [LE(x^2+2x)]_{\delta}$. Because $x^2 - 2x = (x^2+x+1) - 3(x+1) + 2(1)$, we see that

$$LE(x^2 - 2x) = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 4 & 13 \\ 1 & 5 & 21 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix}.$$

Since $LE(f(x)) = (f(2), f(3), f(4))$, we can check

$$LE(x^2 - 2x) = \begin{pmatrix} (2)^2 - 2(2) \\ (3)^2 - 2(3) \\ (4)^2 - 2(4) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix}.$$