Worksheet for April 7

MATH 24 — SPRING 2014

Sample Solutions

(A) For each of the following linear transformations T_k: (i) find a basis for the null space $\mathsf{N}(T_k)$, (ii) extend that basis to the whole domain space, (iii) find a basis for the range $\mathsf{R}(T_k)$,.

1.– $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ where $T_1(a_1, a_2) = (a_1 + a_2, a_1 - a_2)$.

2.– $T_2 : F^3 \to F^2$ where $T_2(a_1, a_2, a_3) = (a_1 - a_2, a_2 - a_3)$.

3.– $T_3 : \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R})$ where $T_3(A) = A - A^t$.

4.– $T_4 : \mathsf{M}_{2\times 2}(\mathbb{R}) \to \mathsf{M}_{2\times 2}(\mathbb{R})$ where $T_4(A) = A + A^t$.

5.– $T_5 : \mathsf{M}_{n\times n}(F) \to F$ where $T_5(A) = \operatorname{tr}(A)$.

Solution —

1.– We have $(a_1, a_2) \in \mathsf{N}(T_1)$ exactly when $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$. This system of equation has only one solution $(a_1, a_2) = (0, 0)$ and therefore $\mathsf{N}(T_1) = \{(0, 0)\}$ and a basis for this space is simply $\varnothing$.

We can extend this to the standard basis $\{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$. In fact, any basis for $\mathbb{R}^2$ will do in this case.

Then, $\{T_1(1, 0), T_1(0, 1)\} = \{(1, 1), (1, -1)\}$ is a basis for $\mathsf{R}(T_1)$. Again, any basis for $\mathbb{R}^2$ will do since $\mathsf{R}(T_1) = \mathbb{R}^2$ for this particular transformation.

2.– We have $(a_1, a_2, a_3) \in \mathsf{N}(T_2)$ exactly when $a_1 - a_2 = 0$ and $a_2 - a_3 = 0$. These two equations combine to $a_1 = a_2 = a_3$, so $\mathsf{N}(T_2) = \operatorname{span}\{(1, 1, 1)\}$ and a basis for the null space of T_2 is $\{(1, 1, 1)\}$.

We can extend this to a basis for F^3 by thinning down the generating set

$$\{(1, 1, 1), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

to a basis containing $(1, 1, 1)$. Proceeding in order, we obtain the basis

$$\{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}.$$

Different choices of bases are also possible, any basis for F^3 that contains $(1, 1, 1)$ will work.

In the previous step, we added two new vectors $(1, 0, 0)$ and $(0, 1, 0)$ to extend our basis. Therefore, $\{T_2(1, 0, 0), T_2(0, 1, 0)\} = \{(1, 0), (1, -1)\}$ is a basis for $\mathsf{R}(T_2)$. Alternatively, we could have first noticed that $\mathsf{R}(T_2) = F^2$ and any basis for F^2 would work for this last step.

3.– Since $A - A^t = 0$ precisely when $A = A^t$, $\mathsf{N}(T_3)$ is the space of symmetric 2×2 matrices, which by Example 19 on pages 50–51 has basis $\{A^{11}, A^{12}, A^{22}\}$, where

$$A^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A^{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This basis can be extended to the basis $\{A^{11}, A^{12}, A^{22}, E^{21}\}$ for $\mathsf{M}_{2\times 2}(\mathbb{R})$. where

$$E^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Finally,

$$\{T_3(E^{21})\} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

is a basis for $\mathsf{R}(T_3)$.

4.– Since $A + A^t = 0$ precisely when $A^t = -A$, $\mathsf{N}(T_4)$ is the space of skew-symmetric 2×2 matrices. After Quiz 1, we know that this space has basis $\{B\}$ where

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This basis can be extended to the basis $\{B, A^{11}, A^{12}, A^{22}\}$ for $\mathsf{M}_{2\times 2}(\mathbb{R})$. Any basis for $\mathsf{M}_{2\times 2}(\mathbb{R})$ that contains B will do. This choice turnsout to be convenient because $T_4(A) = 2A$ for every 2×2 symmetric matrix A. Since A^{11}, A^{12}, A^{22} are all symmetric, it follows that $\{A^{11}, A^{12}, A^{22}\}$ is a basis for $\mathsf{R}(T_4)$.

5.– $\mathsf{N}(T_5)$ is the space of all $n \times n$ matrices with trace zero. A basis for this space consists of the $n^2 - 1$ matrices

$$\{E^{11} - E^{ii} : 2 \leq i \leq n\} \cup \{E^{ij} : 1 \leq i, j \leq n, i \neq j\}.$$

This basis can be extended to the basis

$$\{E^{11}\} \cup \{E^{11} - E^{ii} : 2 \leq i \leq n\} \cup \{E^{ij} : 1 \leq i, j \leq n, i \neq j\}$$

for $\mathsf{M}_{n\times n}(F)$. In fact, any $n \times n$ matrix with nonzero trace would do instead of E^{11} since we know that a basis must have size n^2.

Finally, $\{T_5(E^{11})\} = \{1\}$ is a basis for $\mathsf{R}(T_5)$. Since the codomain F is a 1-dimensional space, there was little choice here.

(B) For which real numbers $b_1, b_2, b_3, b_4, b_5, b_6$ is there a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ such that

$$\begin{aligned} T(1,-1,0,0) &= b_1, & T(0,1,-1,0) &= b_2, \\ T(1,0,-1,0) &= b_3, & T(0,1,0,-1) &= b_4, \\ T(1,0,0,-1) &= b_5, & T(0,0,1,-1) &= b_6? \end{aligned}$$

Solution — The six input vectors are not linearly independent. In particular,

$$(1,0,0,-1) = (0,1,0,-1) + (1,-1,0,0),$$
$$(0,0,1,-1) = (1,0,0,-1) - (1,0,-1,0).$$

So we must have

$$b_5 = T(1,0,0,-1) = T(0,1,0,-1) + T(1,-1,0,0) = b_4 + b_1,$$
$$b_6 = T(0,0,1,-1) = T(1,0,0,-1) - T(1,0,-1,0) = b_5 - b_3.$$

There are no other restrictions on $b_1, b_2, b_3, b_4, b_5, b_6$.

Indeed, the four vectors

$$(1,-1,0,0), (0,1,-1,0), (1,0,-1,0), (0,1,0,-1)$$

form a basis for $\mathbb{R}^4$. According to Theorem 2.6, for any choice of b_1, b_2, b_3, b_4 there is a unique linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ such that

$$T(1,-1,0,0) = b_1, \quad T(0,1,-1,0) = b_2,$$
$$T(1,0,-1,0) = b_3, \quad T(0,1,0,-1) = b_4.$$

So long as $b_5 = b_4 + b_1$ and $b_6 = b_5 - b_3$, this linear transformation will necessarily satisfy

$$T(1,0,0,-1) = b_5, \quad T(0,0,1,-1) = b_6.$$

(C) Let V and W be vector spaces over F. Given a function $T : \mathsf{V} \to \mathsf{W}$, show that the following are equivalent:

1.– T is a linear transformation.

2.– $T(ax + by) = aT(x) + bT(y)$ for all scalars a, b and all vectors $x, y \in \mathsf{V}$.

3.– $T(ax + y) = aT(x) + T(y)$ for every scalar a and all vectors $x, y \in \mathsf{V}$.

Solution — We prove that 1 implies 2, 2 implies 3, and 3 implies 1. Because implication is transitive, this is enough to show that the three requirements are equivalent.

$(1 \Rightarrow 2)$ Suppose a, b are scalars and x, y are vectors in V. Assuming $T : \mathsf{V} \to \mathsf{W}$ is a linear transformation, we have

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

by successively applying properties (a) and (b) of the definition on page 65.

$(2 \Rightarrow 3)$ Suppose $T : \mathsf{V} \to \mathsf{W}$ satisfies condition 3. Choosing $b = 1$ in condition 3, we obtain that

$$T(ax + y) = T(ax + 1y) = aT(x) + 1T(y) = aT(x) + T(y)$$

for all vectors $x, y \in \mathsf{V}$ and every scalar a.

$(3 \Rightarrow 1)$ Suppose $T : \mathsf{V} \to \mathsf{W}$ satisfies condition 3. Choosing $a = 1$ in condition 3, we obtain that

$$T(x + y) = T(1x + y) = 1T(x) + T(y) = T(x) + T(y)$$

for all vectors $x, y \in \mathsf{V}$.

It follows from this that $T(0) = 0$. Indeed,

$$T(0) = T(0 + 0) = T(0) + T(0)$$

and then adding $-T(0)$ to both sides, we obtain $0 = T(0)$.

Choosing $y = 0$ in condition 3, we obtain that

$$T(ax) = T(ax + 0) = aT(x) + T(0) = aT(x) + 0 = aT(x)$$

for every scalar a and every vector x in V.

Since $T(x + y) = T(x) + T(y)$ and $T(ax) = aT(x)$, we conclude that T is a linear transformation.