Worksheet for April 4

MATH 24 — SPRING 2014

Sample Solutions

The following theorems form the skeleton of an alternate development of the key results of Section 1.6 on bases and dimension.

THEOREM 1. *If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V such that*

$$v_i \notin \operatorname{span}\{v_1, \ldots, v_{i-1}\}$$

for $i = 1, 2, \ldots, k$, then the set $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

Proof. We prove this indirectly: assuming that $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent we will show that $v_i \in \operatorname{span}\{v_1, \ldots, v_{i-1}\}$ for some $i \in \{1, 2, \ldots, k\}$.

Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_i v_i = 0$$

where $a_i \neq 0$. Since $a_i \neq 0$, we can solve for v_i above to obtain:

$$v_i = -\frac{a_1}{a_i} v_1 - \frac{a_2}{a_i} v_2 - \cdots - \frac{a_{i-1}}{a_i} v_{i-1}.$$

Therefore, $v_i \in \operatorname{span}\{v_1, v_2, \ldots, v_{i-1}\}$.

We thus conclude that if $v_i \notin \operatorname{span}\{v_1, \ldots, v_{i-1}\}$ for every $i \in \{1, 2, \ldots, k\}$ then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. □

THEOREM 2. *Suppose A is a finite set of vectors in a vector space V. If $C \subseteq A$ is linearly independent then there is a linearly independent set B such that $C \subseteq B \subseteq A$ and $\operatorname{span}(B) = \operatorname{span}(A)$.*

Proof. For any fixed set C of i linearly independent vectors in V, we will prove the result by induction on $k \geq i$, where k is the number of vectors in the set A.

Base Case ($k = i$): Then $A = C$, so choosing $B = A = C$ always meets the requirements of the theorem.

- $C \subseteq B \subseteq A$,
- B is linearly independent since C is and $B = C$, and
- $\operatorname{span}(B) = \operatorname{span}(A)$ since $B = A$.

Induction Step ($k \to k+1$): Let $A = \{v_1, v_2, \ldots, v_k, v_{k+1}\}$ be a given set of $k+1$ vectors from V, where $C = \{v_1, v_2, \ldots, v_i\}$. The induction hypothesis for the set $A_0 = \{v_1, v_2, \ldots, v_k\}$ of k vectors tells us that there is a set B_0 such that:

- $C \subseteq B_0 \subseteq A_0$,
- B_0 is linearly independent, and
- $\operatorname{span}(B_0) = \operatorname{span}(A_0)$.

We now consider two cases depending on whether or not $v_{k+1} \in \operatorname{span}(B_0)$.

In the case where $v_{k+1} \in \operatorname{span}(B_0)$, the set $B = B_0$ works:

- $C \subseteq B \subseteq A$ because $B = B_0 \subseteq A_0 \subseteq A$,
- B is linearly independent because B_0 is, and
- $\operatorname{span}(B) = \operatorname{span}(A)$ by Theorem 1.5 since $A = A_0 \cup \{v_{k+1}\} \subseteq \operatorname{span}(B)$.

In the case where $v_{k+1} \notin \operatorname{span}(B_0)$, the set $B = B_0 \cup \{v_{k+1}\}$ works:

- $C \subseteq B \subseteq A$ because $B = B_0 \cup \{v_{k+1}\} \subseteq A_0 \cup \{v_{k+1}\} = A$,
- B is linearly independent by Theorem 1.5 because B_0 is linearly independent and $v_{k+1} \notin \operatorname{span}(B_0)$, and
- $\operatorname{span}(B) = \operatorname{span}(A)$ by Theorem 1.5 since $A = A_0 \cup \{v_{k+1}\} \subseteq \operatorname{span}(B)$.

Either way, we found a suitable set B. We can therefore conclude that the result is true when the set A has $k+1$ elements.

By the principle of mathematical induction, we conclude that the result is true for every finite set A of vectors containing C. Since C was an arbitrary finite linearly independent subset of V, we conclude that the result is true for all suitable A and C. □

THEOREM 3. *Every finite generating set in a vector space* V *contains a basis for* V.

Proof. Suppose A is a finite generating set of vectors for V.

By Theorem 2, there is a set B such that:

1. $\varnothing \subseteq B \subseteq A$,
2. B is linearly independent, and
3. $\operatorname{span}(B) = \operatorname{span}(A) = \mathsf{V}$.

Thus, B is a basis for V contained in A. □

THEOREM 4. *Every finite linearly independent set in a finitely generated vector space* V *can be extended to a basis for* V.

Proof. Suppose C is a finite linearly independent subset of V and that A is a finite generating set for V. We may further assume that $C \subseteq A$, otherwise replace A with the larger finite generating set $A \cup C$.

By Theorem 2, there is a set B such that:

1. $C \subseteq B \subseteq A$,
2. B is linearly independent, and
3. $\operatorname{span}(B) = \operatorname{span}(A)$.

Since $\operatorname{span}(A) = \mathsf{V}$ by hypothesis, we conclude that B is a basis for V extending C. □

THEOREM 5. *If $v_1, v_2, \ldots, v_k$ is a finite list of vectors in a vector space V then every list of $k+1$ (or more) vectors from $\operatorname{span}\{v_1, v_2, \ldots, v_k\}$ is linearly dependent.*

Proof. We proceed by induction on $k \geq 1$.

Base Case ($k = 1$): Suppose $x_1, x_2 \in \operatorname{span}\{v_1\}$, say $x_1 = a_1 v_1$ and $x_2 = a_2 v_1$. Then

$$a_2 x_1 - a_1 x_2 = a_2(a_1 v_1) - a_1(a_2 v_1) = (a_2 a_1 - a_1 a_2) v_1 = 0 v_1 = 0.$$

On the one hand, if $a_1 \neq 0$ or $a_2 \neq 0$, this shows that x_1, x_2 are linearly dependent. On the other hand, if $a_1 = a_2 = 0$ then $x_1 = x_2 = 0$ and hence x_1, x_2 are again linearly dependent since any list containing the zero vector is linearly dependent.

Induction Step ($k - 1 \to k$): Suppose $x_1, x_2, \ldots, x_k, x_{k+1} \in \operatorname{span}\{v_1, v_2, \ldots, v_k\}$, say:

$$\begin{aligned}
x_1 &= a_{1,1} v_1 + a_{1,2} v_2 + \cdots + a_{1,k} v_k \\
x_2 &= a_{2,1} v_1 + a_{2,2} v_2 + \cdots + a_{2,k} v_k \\
&\;\;\vdots \\
x_k &= a_{k,1} v_1 + a_{k,2} v_2 + \cdots + a_{k,k} v_k \\
x_{k+1} &= a_{k+1,1} v_1 + a_{k+1,2} v_2 + \cdots + a_{k+1,k} v_k
\end{aligned}$$

We consider two cases.

In the case where $a_{1,k} = a_{2,k} = \cdots = a_{k,k} = a_{k+1,k} = 0$, then we actually have that $x_1, x_2, \ldots, x_k, x_{k+1} \in \operatorname{span}\{v_1, v_2, \ldots, v_{k-1}\}$. Therefore, the induction hypothesis applies directly to conclude that $x_1, x_2, \ldots, x_k, x_{k+1}$ are linearly dependent.

Otherwise, at least one of the $a_{i,k}$ is nonzero. Without loss of generality, we may assume $a_{k+1,k} \neq 0$. Consider the vectors $y_1, y_2, \ldots, y_k$ defined by

$$y_1 = x_1 - \frac{a_{1,k}}{a_{k+1,k}} x_{k+1}, \quad y_2 = x_2 - \frac{a_{2,k}}{a_{k+1,k}} x_{k+1}, \quad \ldots, \quad y_k = x_k - \frac{a_{k,k}}{a_{k+1,k}} x_{k+1}.$$

Observe that $y_1, y_2, \ldots, y_k \in \operatorname{span}\{v_1, v_2, \ldots, v_{k-1}\}$. Therefore the induction hypothesis applies to these vectors to conclude that there are scalars $b_1, b_2, \ldots, b_k$, not all zero, such that

$$0 = b_1 y_1 + b_2 y_2 + \cdots + b_k y_k.$$

Substituting $y_i = x_i - \frac{a_{i,k}}{a_{k+1,k}} x_{k+1}$, we find

$$0 = b_1 x_1 + b_2 x_2 + \cdots + b_k x_k - c x_{k+1},$$

where

$$c = b_1 \frac{a_{1,k}}{a_{k+1,k}} + b_2 \frac{a_{2,k}}{a_{k+1,k}} + \cdots + b_k \frac{a_{k,k}}{a_{k+1,k}}.$$

Since the scalars $b_1, b_2, \ldots, b_k$ are not all zero, this shows that the vectors $x_1, x_2, \ldots, x_k, x_{k+1}$ are linearly dependent.

By the principle of mathematical induction, we conclude that the result is true for all finite list of vectors $v_1, v_2, \ldots, v_k$. □

THEOREM 6. *If A and B are finite subsets of a vector space V such that A generates V and B is linearly independent, then A contains at least as many vectors as B.*

Proof. Suppose A and B are finite subsets of a vector space V. We will prove the desired result indirectly: assuming that A generates V and that A has fewer vectors than B, we will show that B is linearly dependent.

Write $A = \{v_1, v_2, \ldots, v_k\}$ and $B = \{x_1, x_2, \ldots, x_\ell\}$, where the two enumerations contain no repetitions. Since B has more vectors than A, it follows that $\ell \geq k + 1$. By Theorem 5, the vectors $x_1, x_2, \ldots, x_{k+1}$ must be linearly dependent. It then follows from Theorem 1.6 that B is linearly dependent too.

We therefore conclude that if A generates V and B is linearly independent, then A contains at least as many vectors as B. □

THEOREM 7. *If the vector space V is finitely generated, then V has a finite basis and all bases for V have the same size.*

Proof. First, the fact that V has a finite basis is a direct consequence of Theorem 3. So it suffices to show that any two finite bases A and B for V must have the same size.

Since A generates V and B is linearly independent, it follows from Theorem 6 that A has at least as many elements as B. Similarly, since B generates V and A is linearly independent, it follows from Theorem 6 that B has at least as many elements as A. Given these two facts, the only possibility is that A and B have the same size. □

DEFINITION. *A vector space V is **finite dimensional** if it has a finite basis. The common size of all the bases for V is called the **dimension** of V and it is often denoted $\dim(\mathsf{V})$.*

THEOREM 8. *Suppose V is a vector space of dimension n.*

(a) *Every linearly independent subset of V with size n is a basis.*

(b) *Every generating set for V with size n is a basis.*

Proof. Assume V has dimension n and suppose that A is a subset of V with size n.

(a) If A is linearly independent, then A can be extended to a basis B for V by Theorem 4. By Theorem 7, B must have size n and therefore $B = A$ since A already has size n. Therefore, A is a basis for V.

(b) If A generates V, then A contains a basis B for V by Theorem 3. By Theorem 7, B must have size n and therefore $B = A$ since A already has size n. Therefore, A is a basis for V. □