Worksheet for March 31

MATH 24 — SPRING 2014

Sample Solutions

(A) How many solutions does each of the systems of equations

$$\begin{aligned} x_1 - x_2 + 5x_3 &= -1 \\ x_2 - 3x_3 &= 2 \\ 2x_1 + x_2 + \ \ x_3 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 3 \\ x_2 - 3x_3 &= -2 \\ 2x_1 + x_2 + \ \ x_3 &= 0 \end{aligned}$$

have over the field $\mathbb{R}$?

Solution — Interestingly, because the left-hand side of the two systems are identical, we can solve them simultaneously using the same operations. Subtract 2 times the first equation from the third to obtain:

$$\begin{aligned} x_1 - \ \ x_2 + 5x_3 &= -1 \\ x_2 - 4x_3 &= 2 \\ 3x_2 - 9x_3 &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 - \ \ x_2 + 5x_3 &= 3 \\ x_2 - 4x_3 &= -2 \\ 3x_2 - 9x_3 &= -6 \end{aligned} \ .$$

Then subtract 3 times the second equation from the third to obtain:

$$\begin{aligned} x_1 - x_2 + 5x_3 &= -1 \\ x_2 - 4x_3 &= 2 \\ 0 &= -3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 - x_2 + 5x_3 &= 3 \\ x_2 - 4x_3 &= -2 \\ 0 &= 0 \end{aligned}$$

We immediately see that the first system is has no solutions since $0 \neq -3$.

We also see that the second system at least has the solution $x_1 = 1$, $x_2 = -2$, and $x_3 = 0$. In fact, it has infinitely many solutions. We can choose any value for x_3, say $x_3 = t$. Then, the second equation forces $x_2 = -2 + 4x_2 = -2 + 4t$. Finally, the first requation forces

$$x_1 = 3 + x_2 - 5x_3 = 3 + (-2 + 4t) - 5t = 1 - t.$$

This process gives a different solution for each choice of real number t. Since there are infinitely many real numbers, there are infinitely many solutions to this system.

(B) Determine whether the vectors $(-1, 2, 1)$ and $(3, -2, 0)$ are in the span of the set

$$S = \{(1, 0, 2), (-1, 1, 1), (5, -3, 1)\}$$

in $\mathbb{R}^3$.

Solution — For the first vector, we are asked whether there are scalars a_1, a_2, a_3 such that

$$a_1(1,0,2) + a_2(-1,1,1) + a_3(5,-3,1) = (-1,2,1).$$

The three coordinate equations lead to the first system from the previous problem, except that the unknowns are now called a_1, a_2, a_3 instead of x_1, x_2, x_3. Since this system has no solutions, we conclude that there are no such scalars a_1, a_2, a_3 and therefore that $(-1,2,1)$ is not in the span of the set S.

Similarly, for the second vector, we are asked whether there are scalars a_1, a_2, a_3 such that

$$a_1(1,0,2) + a_2(-1,1,1) + a_3(5,-3,1) = (3,-2,0).$$

The three coordinate equations lead to the first system from the previous problem, except that the unknowns are now called a_1, a_2, a_3 instead of x_1, x_2, x_3. This system is consistent and it has the solution $a_1 = 1, a_2 = -2, a_3 = 0$. So $(3,-2,0)$ is in the span of S and, in fact,

$$(1,0,2) - 2(-1,1,1) = (3,-2,0)$$

is one way to write this vector as a linear combination of elements of S.

(C) Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generate $\mathsf{M}_{2\times 2}(\mathbb{R})$.

Solution — We are asked to show that every 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

can be written as a linear combination of the four matrices listed above. Well,

$$A = \frac{a_{11}+a_{22}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a_{11}-a_{22}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_{21}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is one way to do just that!

While it's not difficult to guess the representation above, a more systematic approach is to set up a system of linear equations as follows. We are looking for scalars c_1, c_2, c_3, c_4 such that

$$A = c_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The four coordinate equations resulting from this are:

$$\begin{aligned} a_{11} &= c_1 + c_2, & a_{12} &= c_3, \\ a_{21} &= c_4, & a_{22} &= c_1 - c_2. \end{aligned}$$

If we solve for c_1, c_2, c_3, c_4, we obtain

$$c_1 = \frac{a_{11} + a_{22}}{2}, \quad c_2 = \frac{a_{11} - a_{22}}{2}, \quad c_3 = a_{12}, \quad c_4 = a_{21},$$

which is exactly the same solution as above. The upshot of taking this longer route is that we realize that the solution we found is actually the *only* way to represent A as a linear combination of the four given matrices.

(D) Show that the subspace of $\mathsf{M}_{2\times 2}(F)$ spanned by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

consists precisely of all 2×2 matrices over F with trace 0.

Solution — I will formulate this solution in theorem-proof style.

Theorem. *The subspace of* $\mathsf{M}_{2\times 2}(F)$ *spanned by the set*

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

is the space of all 2×2 *matrices over* F *with trace* 0.

Proof. Let W denote the subspace of $\mathsf{M}_{2\times 2}(F)$ consisting of all matrices with trace 0, i.e.,

$$\mathsf{W} = \{A \in \mathsf{M}_{2\times 2}(F) : \operatorname{tr}(A) = 0\}.$$

(We know that this is a subspace of $\mathsf{M}_{2\times 2}(F)$ by Example 4 of Section 1.3.)

By inspection, all three matrices in S have trace 0. Therefore $S \subseteq \mathsf{W}$ and hence, by Theorem 1.5 (more precisely the second sentence thereof), $\operatorname{span}(S) \subseteq \mathsf{W}$. Thus, in order to prove that $\operatorname{span}(S) = \mathsf{W}$, it suffices to show that $\mathsf{W} \subseteq \operatorname{span}(S)$. That is, it suffices to show that every 2×2 matrix with trace 0 is a linear combination of matrices in the set S.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a matrix in W. Since $\operatorname{tr}(A) = a + d$, we must have $a + d = 0$ or, equivalently, $d = -a$. Now

$$a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = A,$$

which shows that A is indeed a linear combination of matrices in the set S.

Since $\operatorname{span}(S) \subseteq \mathsf{W}$ and $\mathsf{W} \subseteq \operatorname{span}(S)$, we conclude that $\operatorname{span}(S) = \mathsf{W}$, i.e., that $\operatorname{span}(S)$ is the space of all 2×2 matrices over F with trace 0. $\square$

(E) Find some polynomials that generate the subspace of $\mathsf{P}_2(\mathbb{R})$ described by the differential equation

$$xf'(x) = f(x).$$

(As in calculus, $f'(x)$ denotes the derivative of the polynomial $f(x)$.)

Solution — From calculus, we know that if

$$f(x) = a_2x^2 + a_1x + a_0$$

then

$$f'(x) = 2a_2x + a_1.$$

So the equation $xf'(x) = f(x)$ can be rewritten:

$$2a_2x^2 + a_1x = a_2x^2 + a_1x + a_0.$$

Matching coefficients of equal degree, we find the three equations

$$2a_2 = a_2, \quad a_1 = a_1, \quad 0 = a_0.$$

The only solutions to these equations are when $a_0 = a_2 = 0$ but a_1 can be any real number. So the solutions of this differential equation is the subspace of $\mathsf{P}_2(\mathbb{R})$ generated by the polynomial x.

Aside: In a differential equations courses like Math 23, you will see that solutions to differential equations can often be described using subspaces. This is because the derivative is a *linear operator*:

$$(f+g)'(x) = f'(x) + g'(x) \quad \text{and} \quad (cf)'(x) = cf'(x),$$

where c is a scalar and f, g are differentiable functions. For example, the solution space of the differential equation

$$f'(x) = f(x)$$

is $\mathrm{span}(e^x)$, which means that the solutions of this equation are precisely the functions of the form $f(x) = ce^x$ where c is an arbitrary constant. Similarly, the solution space of the differential equation

$$f''(x) + f(x) = 0$$

is $\mathrm{span}(\sin(x), \cos(x))$, which means that the solutions of this equation are precisely the functions of the form $f(x) = a\sin(x) + b\cos(x)$ where a and b are arbitrary constants.

What vector space are these solution spaces subspaces of?