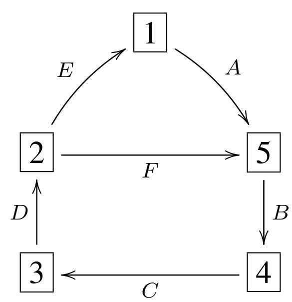
Worksheet for Woozle Flows

MATH 24 — SPRING 2014

Sample Solutions

Woozles are traveling in the maze illustrated below:



Each path is labeled with an element of the set $P = \{A, B, C, D, E, F\}$. Sensors are placed on each path to allow you to measure the flow of woozles along the path. Such measurements give you a function $f \in \mathcal{F}(P, \mathbb{R})$. The value $f(A)$ gives you the flow of woozles along path A, a positive value indicating a flow along the arrow and a negative value indicating flow against the arrow, and similarly for the other paths B, C, D, E, F. The space $\mathcal{F}(P, \mathbb{R})$ has the standard basis $\{e_A, e_B, e_C, e_D, e_E, e_F\}$, where

$$e_Z(X) = \begin{cases} 1 & \text{when } X = Z, \\ 0 & \text{when } X \neq Z, \end{cases}$$

for $Z = A, B, C, D, E, F$. Thus,

$$f = f(A)e_A + f(B)e_B + f(C)e_C + f(D)e_D + f(E)e_E + f(F)e_F$$

for every $f \in \mathcal{F}(P, \mathbb{R})$.

(A) Show that there is a unique linear transformation $T : \mathcal{F}(P, \mathbb{R}) \to \mathbb{R}^5$ such that:

$$T(e_A) = (-1, 0, 0, 0, 1), \quad T(e_B) = (0, 0, 0, 1, -1), \quad T(e_C) = (0, 0, 1, -1, 0),$$
$$T(e_D) = (0, 1, -1, 0, 0), \quad T(e_E) = (1, -1, 0, 0, 0), \quad T(e_F) = (0, -1, 0, 0, 1).$$

Solution — Theorem 2.6 ensures the existence of a unique such transformation. The proof of Theorem 2.6, interpreted in the context at hand, gives us a more convenient way to think about this linear transformation T. Indeed, we can compute $T(f)$ as follows:

$$
\begin{aligned}
T(f) &= T(f(A)e_A + f(B)e_B + f(C)e_C + f(D)e_D + f(E)e_E + f(F)e_F) \\
&= f(A)T(e_A) + f(B)T(e_B) + f(C)T(e_C) + f(D)T(e_D) + f(E)(e_E) + f(F)T(e_F) \\
&= f(A)\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + f(B)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} + f(C)\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + f(D)\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + f(E)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + f(F)\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -f(A) + f(E) \\ f(D) - f(E) - f(F) \\ f(C) - f(D) \\ f(B) - f(C) \\ f(A) - f(B) + f(F) \end{pmatrix}
\end{aligned}
$$

This is a more convenient way to think about T since it gives us a way to compute each coordinate of the output in terms of the values of the input function f.

(B) Find a basis for the null space $\mathsf{N}(T)$. For each vector f in your basis, draw a picture of the maze where each path Z is labeled with the flow $f(Z)$.

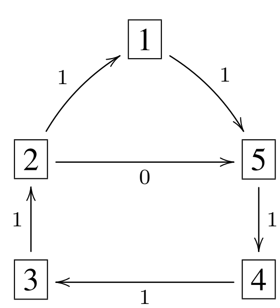
Solution — Looking at the representation of T we obtained in part (A) and setting each coordinate equal to zero, we obtain the following system of linear equations:

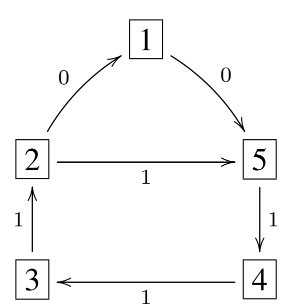
$$
\begin{array}{rrrrrrl}
-f(A) & & & & + f(E) & & = 0 \\
 & & & f(D) & - f(E) & - f(F) & = 0 \\
 & & f(C) & - f(D) & & & = 0 \\
 & f(B) & - f(C) & & & & = 0 \\
f(A) & - f(B) & & & & + f(F) & = 0
\end{array}
$$

Reorganizing into echelon form using the method of Section 1.4, we obtain:

$$
\begin{array}{rrrrrrl}
f(A) & & & & - f(E) & & = 0 \\
 & f(B) & - f(C) & & & & = 0 \\
 & & f(C) & - f(D) & & & = 0 \\
 & & & f(D) & - f(E) & - f(F) & = 0
\end{array}
$$

(The last row ends up being $0 = 0$, so it was omitted.) We have two slack variables, $f(E)$ and $f(F)$. Setting $f_1(E) = 1, f_1(F) = 0$ and $f_2(E) = 0, f_2(F) = 1$, we obtain the two solutions depicted below, respectively:





Since every solution f to the above of system of equations must satisfy $f = f(E)f_1 + f(F)f_2$, these two vectors generate $\mathsf{N}(T)$. Since f_1, f_2 are visibly not scalar multiples of each other, they are linearly independent. We therefore conclude that $\{f_1, f_2\}$ is a basis for $\mathsf{N}(T)$.

The first function f_1 corresponds to one woozle running in a loop around the circumference of the maze and the second function f_2 corresponds to one woozle running in a loop around the bottom square of the maze. Observe that one woozle running in a loop around the top triangle of the maze corresponds to the linear combination $f_1 - f_2$. What do you think general elements of $\mathsf{N}(T)$ represent?

(C) Suppose a heffalump sits in the middle of path A, blocking the flow of woozles along this path. Describe the subspace W of $\mathcal{F}(P, \mathbb{R})$ corresponding to the possible measurements you could make after this event.

Solution — Since the flow along path A must be zero, and this is the only restriction,

$$\mathsf{W} = \{f \in \mathcal{F}(P, \mathbb{R}) : f(A) = 0\}.$$

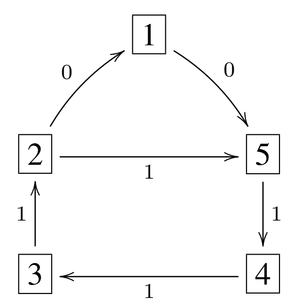
Note that you have shown that such a W is always a subspace of $\mathcal{F}(P, \mathbb{R})$ in Exercise 13 of Section 1.3.

(D) Find a basis for $\mathsf{N}(T) \cap \mathsf{W}$. For each vector f in your basis, draw a picture of the maze where each path Z is labeled with the flow $f(Z)$.

Solution — Since $\mathsf{N}(T) \cap \mathsf{W}$ consists of all functions $f \in \mathcal{F}(P, \mathbb{R})$ that are in both subspaces, the restrictions on such f lead to the system of linear equations from part (B) along with the additional equation $f(A) = 0$ from part (B). Once put into echelon form, we obtain the system

$$\begin{aligned}
f(A) \qquad\qquad\qquad\qquad - f(E) \qquad\quad &= 0\\
f(B) - f(C) \qquad\qquad\qquad\qquad &= 0\\
f(C) - f(D) \qquad\qquad &= 0\\
f(D) - f(E) - f(F) &= 0\\
f(E) \qquad\qquad &= 0
\end{aligned}$$

This new system now has only one slack variable, $f(F)$. Setting $f(F) = 1$, we obtain the solution f_2 from part (B):



Any solution f to the system above must satisfy $f = f(F)f_2$ and since $f_2 \neq 0$ we conclude that $\{f_2\}$ is a basis for $\mathsf{N}(T) \cap \mathsf{W}$.

This makes sense since the bottom square of the maze is the only closed loop left after removing path A.

Interpret and extrapolate your findings.

- Can you explain in words what the meaning of the output of T is? Is there a relation between the coordinates of the output and the numerical labels in the maze?
- Can you explain why elements of the null space $\mathsf{N}(T)$ seem to correspond to closed loops in the maze? What exactly is meant by 'closed loop' here?
- Can you reason what would happen in part (D) if the heffalump had sat on path C or path F instead of path A? What if two heffalumps sat on both paths C and F?
- Can you think of a way to interpret the range space $\mathsf{R}(T)$? Can you explain why the range is not all of $\mathbb{R}^5$?