Slides for May 7

Math 24 — Spring 2014

Eigenspaces

Definition

Given linear operator $T : \mathsf{V} \to \mathsf{V}$ and a scalar λ, we define

$$\mathsf{E}_\lambda = \mathsf{N}(T - \lambda I).$$

If λ is an eigenvalue of T then E_λ is the **eigenspace** of T associated to λ.

- The scalar λ is an eigenvalue of T if and only if $\mathsf{E}_\lambda \neq \{0\}$.
- The eigenvectors of T correponding to the eigenvalue λ are precisely the nonzero elements of E_λ.

Eigenspace Decomposition

Theorem

Suppose $T : \mathsf{V} \to \mathsf{V}$ *is a linear operator on an n-dimensional vector space* V *and* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are* *__all__* *the eigenvalues of* T*, without repetitions. The following are equivalent:*

(1) T *is diagonalizable.*

(2) T *has a basis of eigenvectors.*

(3) $\dim(\mathsf{E}_{\lambda_1}) + \dim(\mathsf{E}_{\lambda_2}) + \cdots + \dim(\mathsf{E}_{\lambda_k}) = n.$

(4) $\mathsf{E}_{\lambda_1} + \mathsf{E}_{\lambda_2} + \cdots + \mathsf{E}_{\lambda_k} = \mathsf{V}.$

Lemma

A vector $x \in \mathsf{V}$ *has at most one decomposition*

$$x = v_1 + v_2 + \cdots + v_k$$

such that $v_1 \in \mathsf{E}_{\lambda_1}, v_2 \in \mathsf{E}_{\lambda_2}, \ldots, v_k \in \mathsf{E}_{\lambda_k}.$

Algebraic and Geometric Multiplicity

Definition

Let $T : \mathrm{V} \to \mathrm{V}$ be a linear operator on a finite dimensional vector space V.

- The **geometric multiplicity** of an eigenvalue λ of T is the dimension of E_λ.
- The **algebraic multiplicity** of an eigenvalue λ of T is the multiplicity of the root λ in the characteristic polynomial $\det(T - tI)$.

Example

If $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ then the algebraic multiplicity of λ is 2 but the geometric multiplicity of λ is 1.

Algebraic and Geometric Multiplicity

Theorem

Let $T : \mathrm{V} \to \mathrm{V}$ be a linear operator over a finite dimensional vector space over the field F. Then T is diagonalizable if and only if both of the following hold:

- *The characteristic polynomial of T splits into linear factors over F.*
- *Every eigenvalue of T has equal geometric and algebraic multiplicities.*

Lemma

The geometric multiplicity of an eigenvalue never exceeds its algebraic multiplicity.