Slides for May 5

MATH 24 — SPRING 2014

Diagonalizability

Definition

A linear operator $T : \mathrm{V} \to \mathrm{V}$ on a finite dimensional vector space V is **diagonalizable** if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Example

If $S : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $S(x, y) = (y, x)$ then, with respect to the ordered basis $\beta = \{(1, 1), (1, -1)\}$ for $\mathbb{R}^2$, we have

$$[S]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

because $S(1, 1) = (1, 1)$ and $S(1, -1) = -(1, -1)$.

Diagonalizability

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Example

If $T : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by $T(x, y) = (-y, x)$ then, with respect to the ordered basis $\beta = \{(1, -i), (1, i)\}$ for $\mathbb{C}^2$, we have

$$[T]_\beta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

because $T(1, -i) = i(1, -i)$ and $T(1, i) = -i(1, i)$.

Eigenvectors and Eigenvalues

Definition

Let $T : \mathrm{V} \to \mathrm{V}$ be a linear operator.

- An **eigenvector** for T is a nonzero vector $v \in \mathrm{V}$ such that $T(v) = \lambda v$ for some scalar λ.
- The scalar λ is then called the **eigenvalue** associated to the eigenvector v.

Example

So every nonzero $v \in \mathrm{N}(T)$ is an eigenvector with eigenvalue 0 since $T(v) = 0 = 0v$.

Example

Every nonzero vector $v \in \mathrm{V}$ is an eigenvector for the identity transformation $I : \mathrm{V} \to \mathrm{V}$ since $I(v) = v = 1v$.

Diagonalizability and Eigenvectors

Theorem

If $\beta = \{v_1, v_2, \ldots, v_n\}$ *is an ordered basis for* V *and* $T : \mathsf{V} \to \mathsf{V}$ *is a linear operator such that*

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

then each v_i *is an eigenvector with eigenvalue* λ_i*.*

Corollary

A linear operator $T : \mathsf{V} \to \mathsf{V}$ *on a finite dimensional vector space* V *is diagonalizable if and only if there is a basis for* V *that consists of eigenvectors for* T*.*

Finding Eigenvectors given Eigenvalues

Theorem

Let $T : \mathsf{V} \to \mathsf{V}$ be a linear operator. Given a scalar λ, the eigenvectors with eigenvalue λ are precisely the <u>nonzero</u> vectors of the null space $\mathsf{N}(T - \lambda I)$.

Given an eigenvalue λ we <u>can</u> find all corresponding eigenvectors

Corollary

Let $T : \mathsf{V} \to \mathsf{V}$ be a linear operator and let λ be a scalar. There exists an eigenvector with eigenvalue λ if and only if $T - \lambda I$ is not invertible.

How can we find <u>all</u> scalars λ such that $T - \lambda I$ is not invertible?

The Characteristic Polynomial of a Square Matrix

Theorem

If A is a $n \times n$ matrix over the field F, then

$$\det(A - tI_n) = (-1)^n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0$$

is a polynomial of degree n in the variable t with coefficients in F.

$\det(A - tI_n)$ is the **characteristic polynomial** of A

Corollary

Let $T : \mathsf{V} \to \mathsf{V}$ be a linear operator and let α be any ordered basis for the finite dimensional vector space V. The eigenvalues of T are the roots of the characteristic polynomial of the matrix $A = [T]_\alpha$.

The Characteristic Polynomial of a Linear Operator

Theorem

Let $T : \mathrm{V} \to \mathrm{V}$ be a linear operator and let α and β be any two ordered bases for the finite dimensional vector space V. Then the matrices $A = [T]_\alpha$ and $B = [T]_\beta$ have the same characteristic polynomial.

Proof.

Let Q be the change of coordinate matrix from α-coordinates to β-coordinates. Then $A = Q^{-1}BQ$ and $tI_n = Q^{-1}(tI_n)Q$, so

$$A - tI_n = Q^{-1}BQ - Q^{-1}(tI_n)Q = Q^{-1}(B - tI_n)Q$$

and hence

$$\det(A - tI_n) = \det(Q^{-1})\det(B - tI_n)\det(Q) = \det(B - tI_n)$$

since $\det(Q^{-1}) = 1/\det(Q)$. □

Examples Revisited

$S : \mathbb{R}^2 \to \mathbb{R}^2$

$S(x, y) = (y, x)$

$$[S]_{\text{std}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\det(S - tI) = t^2 - 1$

Eigenvalues: -1, 1

Eigenbasis: $\{(1, -1), (1, 1)\}$

$$[S]_{\text{eig}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$T : \mathbb{C}^2 \to \mathbb{C}^2$

$T(x, y) = (-y, x)$

$$[T]_{\text{std}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\det(T - tI) = t^2 + 1$

Eigenvalues: i, $-i$

Eigenbasis: $\{(1, -i), (1, i)\}$

$$[T]_{\text{eig}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$